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Estimation of Item Parameters in Graded Response Model

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Abstract: *The research and application of item response theory (IRT) models in modern educational examination have widely been paid attention to. However, item parameter estimate plays a key role in applying IRT to practical work, and it is a complex and scientific task. In this present paper, we obtain a method for estimating the parameters* $a_i, b_{u_i}(u_i = 0, 1, \ldots, m_i)$ *of graded response model. It is convenient and adequate when the sample size is small and the requirement of the precision of estimation is not so strict. Well, the parameters we have obtained by this method can be used as the initial value of joint maximum likelihood estimation.*

Keywords: *Item Response Theory, Graded Response Model, Item Parameter Estimation*

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I. INTRODUCTION

1.1 Background

Item response theory (IRT) is also sometimes called latent trait theory, which refers to a set of mathematical models that describe in probabilistic items. IRT involves the relationships between a person's response to a survey question and his or her level of the latent variable being measured by scale ([1]-[9]).

IRT is popular because it provides a theoretical justification that traditional approaches(i.e. classical test theory) do not. For example, traditional approaches give measurement scales which are based on averages or simple summations of the multiple items. In contrast, IRT models are based on the probability that a person will make a particular response according to their level of the underlying latent variable. Therefore, IRT comprises a set of non-linear latent variable models that appear to have several conceptual and empirical properties that make them more valuable in practice.

1.2 Model (Graded Response Model)

As we know, the graded response model is a kind of famous model in IRT models. The graded response model is firstly given by Samejima in 1969, and it is used for dealing with ordered polychotomously-scored items.

Now, we will introduce the graded response model.

Let θ be the examinee's latent trait, and U_i be a random variable sequence as the score of the partial response to the i-th item. And assume that, $u_i(u_i = 0, 1, 2, \ldots, m_i)$ is the record of the practical response, and is the probability of scoring u_i on the ith $P_{u_i}(\theta)$ item for a person who is with ability θ. Usually, the probability $P_{u_i}(\theta)$ is called the category response function. If let $P^*_{u_i}(\theta)$ be as the probability of scoring u_i or topping u_i on the i-th item, then we have

$$P_{u_i}(\theta) = P^*_{u_i}(\theta) - P^*_{u_i+1}(\theta)$$

Suppose that, the examinee who scores u_i or tops u_i is "pass" or "1" on the item i. If one scores less than u_i, then it means that this person is "fail" or his/her score is "0". So, $P^*_{u_i}(\theta)$ just becomes the item characteristic function of dichotomous-scored items.

Here, we note $P^*_0(\theta) = 1$, $P^*_{m_i+1}(\theta) = 0$. Then, we have

$$\begin{cases} P_0^*(\theta) & = & 1 \\ P_1^*(\theta) & = & \dfrac{1}{1+e^{-D\cdot a_i(\theta-b_1)}} \\ \cdots & & \\ P_{m_i}^*(\theta) & = & \dfrac{1}{1+e^{-D\cdot a_i(\theta-b_{m_i})}} \\ P_{m_i+1}^*(\theta) & = & 0 \end{cases}$$

So, $P_{u_i}(\theta)$ can be wrote as

$$\begin{cases} P_0(\theta) & = & 1-P_1^*(\theta) \\ P_1(\theta) & = & P_1^*(\theta)-P_2^*(\theta) \\ \cdots & & \\ P_{m_i}(\theta) & = & P_{m_i}^*(\theta)-P_{m_i+1}^*(\theta)=P_{m_i}^*(\theta) \end{cases} \quad (1)$$

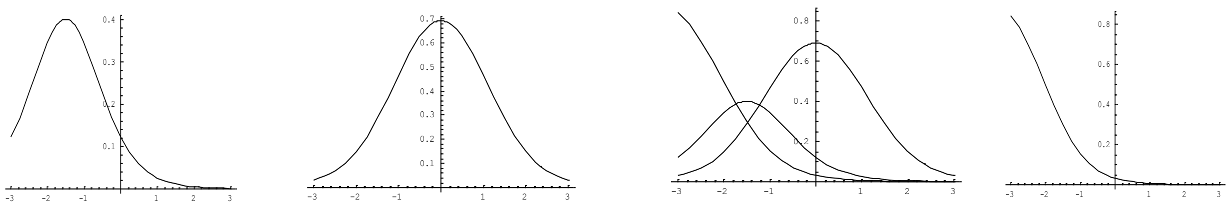
where a_i is the discrimination parameter of the i-th item, and $b_{u_i}(u_i=0,1,\ldots,m_i)$ is the difficulty parameter of the u_i-th grade. When we take D as 1.702, the item characteristic curve (ICC) is scaled in the same metric as the normal ogive ICC. Then, we obtain the anther form of $P_{u_i}(\theta)$, i.e.,

$$P_{u_i}(\theta)=\frac{1}{\sqrt{2\pi}}\cdot\int_{a_i\cdot(\theta-b_{u_i+1})}^{a_i\cdot(\theta-b_{u_i})} e^{-\frac{t^2}{2}}dt, u_i=0,1,\cdots,m_i \quad (2)$$

Here, the graded item response model we discuss has the same a_i on every item, and $-\infty=b_0<b_1<b_2<\cdots<b_{m_i}<b_{m_i+1}=+\infty$.

1.3 Parameters in IRT models

The ICC of the graded response model can be simulated as follows by Mathematica as follows (here, let $m_i=3, a_i=1.0, b_1=-2.0, b_2=-1, b_3=1$).



(1) 1st grade **(2) 2nd grade** **(3) 3th grade**
(4) Comparison

Fig. 1. The ICC of the graded response model.

As in the above graphs, we can see that, ICC enables us to understand that how the examinee's response relates to his or her level of the underlying latent variable. In fact, the parameters of ICC are invariant across populations. In other words, if you pick different samples to estimate ICC, you should get the same values of parameters. That is, you always get the same ICC, and it is a big advantage of IRT: it shows the invariance of item parameters and ability parameters. However, parameter estimation is very critical in IRT, and is also a quite hard task ([1]-[2]).

II. RELATED WORKS

Till now, a lot of statistical methods for estimating parameters have been proposed. For instance, condition maximum likelihood estimation(CMLE), joint maximum likelihood estimation(JMLE), marginal maximum likelihood estimation and EM algorithm(MMLE), marginalized Bayesian estimation and so on ([10]-[13]). However, there exists some drawbacks in these current methods:

1) It usually gets local convergence in estimation, and is difficult to obtain the global maximum;

2) While iterating to estimate the maximum, it is often oscillatory;

Note 1 In practice, reasons for the above phenomenon 1) and 2) are as follows:

There are many parameters in IRT models, but the size of samples is always small. And the process of estimating is greatly depended on the initial value.

Because of the above-mentioned problems, it is necessary to find an effective approximate approach.

III. MOTIVATION

Chen in [14] proposed a method of parameter estimation for two-parameter Logistic model, which is convenience and suitable when the sample size is not large and precision of estimation is not high. Inspired by this method, we proposed a method of obtaining estimations of parameters $a_i, b_{u_i}(u_i = 0,1,\cdots,m_i)$ for graded response model. It is convenient and adequate, when the sample size is small and the necessary for precision of estimation is not so strict. Well, the parameters we have obtained by this way can also be used as the initial value of joint maximum likelihood estimation.

IV. MAIN RESULTS

Let θ be the ability parameter and suppose $\theta \sim N(0,1)$. Let A_{u_i} be the expectation of the difficulty of the u_i-th grade on the i-th item. For the simplicity, we use (2) instead of (1) in the following Theorem 4.2 and Theorem 4.3.

Firstly, we give a lemma for the later discussion.

Lemma 4.1 In graded response model, a_i is the discrimination parameter of the i-th item. Then, we have $a_i \geq 0$.

Now, we give the main theorems in this paper.

Theorem 4.2 If $\theta \sim N(0,1)$, $P_{u_i}(\theta) = (1/\sqrt{2\pi}) \cdot \int_{a_i(\theta - b_{u_i+1})}^{a_i(\theta - b_{u_i})} e^{-\frac{t^2}{2}} dt$. Then, we have

$$\begin{cases} b_1 = \dfrac{\Phi^{-1}(A_0)}{W_i} = \dfrac{\Phi^{-1}(1 - A_{m_i} - A_{m_i-1} - \cdots - A_1)}{W_i} \\ b_2 = \dfrac{\Phi^{-1}(A_1 + A_0)}{W_i} = \dfrac{\Phi^{-1}(1 - A_{m_i} - A_{m_i-1} - \cdots - A_2)}{W_i} \\ \cdots \\ b_{m_i} = \dfrac{\Phi^{-1}(A_{m_i-1} + A_{m_i-2})}{W_i} = \dfrac{\Phi^{-1}(1 - A_{m_i})}{W_i} \end{cases}$$

where $W_i = a_i / \sqrt{1 + a_i^2}$.

Theorem 4.3 Let V_{u_i} be the expectation of the examinee's ability who scores u_i on the i-th item. Then, we have

$$W_i = \frac{\sqrt{2\pi} \cdot V_1 \cdot A_1}{e^{-\frac{k_1^2}{2}} - e^{-\frac{k_2^2}{2}}} = \frac{\sqrt{2\pi} \cdot V_{u_i} \cdot A_{u_i}}{e^{-\frac{k_{u_i}^2}{2}} - e^{-\frac{k_{u_i+1}^2}{2}}}, u_i = 2, 3, \cdots, m_i.$$

where $k_1 = \Phi^{-1}(A_0)$ or $\Phi^{-1}(1 - A_{m_i} - A_{m_i-1} - \cdots - A_1)$;

$k_2 = \Phi^{-1}(A_1 + A_0)$ or $\Phi^{-1}(1 - A_{m_i} - A_{m_i-1} - \cdots - A_2)$;

$k_{u_i} = \Phi^{-1}(A_{u_i} + A_{u_i-1})$ or $\Phi^{-1}(1 - A_{m_i} - A_{m_i-1} - \cdots - A_{u_i}), u_i = 2, 3, \cdots, m_i$.

V. PROOFS

In this section, we will prove Lemma 4.1, Theorems 4.2 and 4.3 in the above section.

The proof of Lemma 4.1

Proof. By the definition of graded response model, we have

$$P_{u_i}(\theta) = \frac{\exp[-D \cdot a_i \cdot (\theta - b_{u_i+1})] - \exp[-D \cdot a_i \cdot (\theta - b_{u_i})]}{\{1 + \exp[-D \cdot a_i \cdot (\theta - b_{u_i})]\}\{1 + \exp[-D \cdot a_i \cdot (\theta - b_{u_i+1})]\}}.$$

Note that

$$P_{u_i}(\theta) \geq 0.$$

So, we have

$$\exp[-D \cdot a_i \cdot (\theta - b_{u_i+1})] - \exp[-D \cdot a_i \cdot (\theta - b_{u_i})] \geq 0.$$

Then

$$a_i \cdot b_{u_i+1} \geq a_i \cdot b_{u_i}.$$

And by $-\infty = b_0 < b_1 < b_2 < \cdots < b_{m_i} < b_{m_i+1} = +\infty$. Therefore,

$$a_i \geq 0.$$

So, we complete the proof.

The proof of Theorem 4.2

Proof.

$$
\begin{aligned}
A_{u_i} &= E[P_{u_i}(\theta)] = \int_{-\infty}^{\infty} P_{u_i}(\theta)\cdot\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{\theta^2}{2}}d\theta \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{\theta^2}{2}}d\theta\int_{a_i(\theta-b_{u_i+1})}^{a_i(\theta-b_{u_i})}\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{t^2}{2}}dt \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{\theta^2}{2}}d\theta[\int_{-\infty}^{a_i(\theta-b_{u_i})}\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{t^2}{2}}dt-\int_{-\infty}^{a_i(\theta-b_{u_i+1})}\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{t^2}{2}}dt] \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{\theta^2}{2}}d\theta\int_{-\infty}^{a_i(\theta-b_{u_i})}\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{t_1^2}{2}}dt_1-\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{\theta^2}{2}}d\theta\int_{-\infty}^{a_i(\theta-b_{u_i+1})}\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{t_2^2}{2}}dt_2 \\
&= \int_{-\infty}^{\infty}\frac{a_i}{\sqrt{2\pi}}\cdot e^{-\frac{\theta^2}{2}}d\theta\int_{\infty}^{b_{u_i}}(-1)\cdot\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{[a_i(\theta-y_1)]^2}{2}}dy_1 \qquad \left(\text{here, let}\quad \begin{matrix} t_1=a_i(\theta-y_1); \\ t_2=a_i(\theta-y_2). \end{matrix}\right) \\
&\quad -\int_{-\infty}^{\infty}\frac{a_i}{\sqrt{2\pi}}\cdot e^{-\frac{\theta^2}{2}}d\theta\int_{\infty}^{b_{u_i+1}}(-1)\cdot\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{[a_i(\theta-y_2)]^2}{2}}dy_2 \\
&= \frac{a_i}{2\pi}\cdot\int_{-\infty}^{\infty}e^{-\frac{\theta^2}{2}}d\theta\int_{b_{u_i}}^{\infty}e^{-\frac{[a_i(\theta-y_1)]^2}{2}}dy_1-\frac{a_i}{2\pi}\cdot\int_{-\infty}^{\infty}e^{-\frac{\theta^2}{2}}d\theta\int_{b_{u_i+1}}^{\infty}e^{-\frac{[a_i(\theta-y_2)]^2}{2}}dy_2 \\
&= \frac{a_i}{2\pi}\cdot\int_{-\infty}^{\infty}\int_{b_{u_i}}^{\infty}e^{-\frac{[(\theta\sqrt{1+a_i^2}-\frac{a_i^2y_1}{\sqrt{1+a_i^2}})^2+a_i^2y_1^2-\frac{a_i4y_1^2}{1+a_i^2}]}{2}}dy_1d\theta-\frac{a_i}{2\pi}\cdot\int_{-\infty}^{\infty}\int_{b_{u_i+1}}^{\infty}e^{-\frac{[(\theta\sqrt{1+a_i^2}-\frac{a_i^2y_2}{\sqrt{1+a_i^2}})^2+a_i^2y_2^2-\frac{a_i4y_2^2}{1+a_i^2}]}{2}}dy_2d\theta \\
&= \frac{a_i}{2\pi}\cdot\frac{1}{\sqrt{1+a_i^2}}\cdot\int_{b_{u_i}}^{\infty}e^{-\frac{(a_i^2y_1^2-\frac{a_i4y_1^2}{1+a_i^2})}{2}}[\int_{-\infty}^{\infty}e^{-\frac{(\theta\sqrt{1+a_i^2}-\frac{a_i^2y_1}{\sqrt{1+a_i^2}})^2}{2}}d\theta\sqrt{1+a_i^2}]dy_1 \\
&\quad -\frac{a_i}{2\pi}\cdot\frac{1}{\sqrt{1+a_i^2}}\cdot\int_{b_{u_i}}^{\infty}e^{-\frac{(a_i^2y_2^2-\frac{a_i4y_2^2}{1+a_i^2})}{2}}[\int_{-\infty}^{\infty}e^{-\frac{(\theta\sqrt{1+a_i^2}-\frac{a_i^2y_2}{\sqrt{1+a_i^2}})^2}{2}}d\theta\sqrt{1+a_i^2}]dy_2 \\
&= \frac{W_i}{\sqrt{2\pi}}\cdot\int_{b_{u_i}}^{\infty}e^{-\frac{W_i^2y_1^2}{2}}dy_1-\frac{W_i}{\sqrt{2\pi}}\cdot\int_{b_{u_i+1}}^{\infty}e^{-\frac{W_i^2y_2^2}{2}}dy_2 \qquad \left(\text{here, let}\quad W_i=\frac{a_i}{\sqrt{1+a_i^2}}.\right) \\
&= 1-\Phi(b_{u_i}\cdot W_i)-(1-\Phi(b_{u_i+1}\cdot W_i)) \\
&= \Phi(b_{u_i+1}\cdot W_i)-\Phi(b_{u_i}\cdot W_i).
\end{aligned}
$$

So, we have

$$A_{u_i} = \Phi(b_{u_i+1}\cdot W_i)-\Phi(b_{u_i}\cdot W_i).$$

Note that, $\Phi(-\infty)=0$, $\Phi(+\infty)=1$. By Lemma 4.1, we get

$$
\begin{cases}
A_0 = \Phi(b_1W_i) \\
A_1 = \Phi(b_2W_i)-\Phi(b_1W_i) \\
\cdots \\
A_{m_i-1} = \Phi(b_{m_i}W_i)-\Phi(b_{m_i-1}W_i) = 1-\Phi(b_{m_i}W_i)
\end{cases}
$$

So, we obtain

$$
\begin{cases}
b_1 = \dfrac{\Phi^{-1}(A_0)}{W_i} = \dfrac{\Phi^{-1}(1-A_{m_i}-A_{m_i-1}-\cdots-A_1)}{W_i} \\
\cdots \\
b_{m_i} = \dfrac{\Phi^{-1}(A_{m_i-1}+A_{m_i-2})}{W_i} = \dfrac{\Phi^{-1}(1-A_{m_i})}{W_i}
\end{cases}
$$

Therefore, we complete the proof.

The proof of Theorem 4.3

Proof. Let

$$x_{u_i} = \begin{cases} 1, \text{Score } u_i \text{ on the ith item;} \\ 0, \text{otherwise.} \end{cases}$$

By the discussion in the former part, we know that the probability of scoring u_i on the i-th item for a person who has ability θ is $P_{u_i}(\theta)$, and let $Q_{u_i} = 1 - P_{u_i}(\theta)$.

Thus, the joint P.D.F. of θ and x_{u_i} is given by

$$f(\theta, x_{u_i}) = (1/\sqrt{2\pi}) \cdot e^{-\frac{\theta^2}{2}} \cdot (P_{u_i}(\theta))^{x_{u_i}} \cdot (Q_{u_i}(\theta))^{1-x_{u_i}}.$$

When $u_i = 1$, we obtain the conditional density of θ as follows:

$$g(\theta \,|\, x_{u_i} = 1) = f(\theta, x_{u_i} = 1) / P(x_{u_i} = 1) = \frac{P_{u_i}(\theta)}{A_{u_i}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}}.$$

So, we get

$$\begin{aligned} & V_{u_i} \\ &= \frac{1}{A_{u_i}} \int_{-\infty}^{\infty} \frac{\theta}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}} \cdot P_{u_i}(\theta) d\theta \\ &= \frac{a_i}{2\pi A_{u_i}} \cdot \int_{-\infty}^{\infty} \int_{b_{u_i}}^{\infty} \theta \cdot e^{-\frac{(\theta\sqrt{1+a_i^2} - \frac{a_i^2 y_1}{\sqrt{1+a_i^2}})^2 + a_i^2 y_1^2 - \frac{a_i^4 y_1^2}{1+a_i^2}}{2}} dy_1 d\theta \\ &\quad - \frac{a_i}{2\pi A_{u_i}} \cdot \int_{-\infty}^{\infty} \int_{b_{u_i+1}}^{\infty} \theta \cdot e^{-\frac{(\theta\sqrt{1+a_i^2} - \frac{a_i^2 y_2}{\sqrt{1+a_i^2}})^2 + a_i^2 y_2^2 - \frac{a_i^4 y_2^2}{1+a_i^2}}{2}} dy_2 d\theta \end{aligned} \tag{3}$$

where $t_1 = a_i(\theta - y_1), t_2 = a_i(\theta - y_2)$.

Let $z_1 = \theta\sqrt{1 + a_i^2} - \frac{a_i^2 y_1}{\sqrt{1 + a_1^2}}$, $z_2 = \theta\sqrt{1 + a_i^2} - \frac{a_i^2 y_2}{\sqrt{1 + a_1^2}}$ and $w_i = a_i / \sqrt{1 + a_i^2}$, then, we have

Equation (3)

$$= \frac{W_i}{2\pi A_{u_i}} \cdot \int_{-\infty}^{\infty}\int_{b_{u_i}}^{\infty} \frac{(z_1 + \frac{a_i^2 y_1}{1+a_i^2})}{\sqrt{1+a_i^2}} e^{-\frac{W_i^2 y_1^2 + z_1^2}{2}} dy_1 dz_1$$

$$- \frac{W_i}{2\pi A_{u_i}} \cdot \int_{-\infty}^{\infty}\int_{b_{u_i+1}}^{\infty} \frac{(z_2 + \frac{a_i^2 y_2}{1+a_i^2})}{\sqrt{1+a_i^2}} e^{-\frac{W_i^2 y_2^2 + z_2^2}{2}} dy_2 dz_2$$

$$= \frac{W_i}{\sqrt{2\pi}\, A_{u_i}} \cdot \int_{b_{u_i}}^{\infty} e^{-\frac{W_i^2 y_1^2}{2}} dy_1 \int_{-\infty}^{\infty} \left(\frac{z_1}{\sqrt{2\pi}\sqrt{1+a_i^2}} + \frac{W_i^2 y_1}{\sqrt{2\pi}}\right) e^{-\frac{z_1^2}{2}} dz_1$$

$$- \frac{W_i}{\sqrt{2\pi}\, A_{u_i}} \cdot \int_{b_{u_{i+1}}}^{\infty} e^{-\frac{W_i^2 y_2^2}{2}} dy_2 \int_{-\infty}^{\infty} \left(\frac{z_2}{\sqrt{2\pi}\sqrt{1+a_i^2}} + \frac{W_i^2 y_2}{\sqrt{2\pi}}\right) e^{-\frac{z_2^2}{2}} dz_2$$

$$= \frac{W_i}{\sqrt{2\pi}\, A_{u_i}} e^{-\frac{W_i^2 b_{u_i}^2}{2}} - \frac{W_i}{\sqrt{2\pi}\, A_{u_i}} e^{-\frac{W_i^2 b_{u_i+1}^2}{2}}$$

where $x_1 = W_i y_1, x_2 = W_i y_2$.

By Theorem 4.2, we see that

$$\begin{cases} b_1 \cdot W_i = \Phi^{-1}(A_0) = \Phi^{-1}(1 - A_{m_i} - A_{m_i-1} \cdots A_1) \\ b_2 \cdot W_i = \Phi^{-1}(A_1 + A_0) = \Phi^{-1}(1 - A_{m_i} - A_{m_i-1} \cdots A_2) \\ \cdots \\ b_{m_i} \cdot W_i = \Phi^{-1}(A_{m_i-1} + A_{m_i-2}) = \Phi^{-1}(1 - A_{m_i}) \end{cases}$$

So, we have

$$V_1 = \frac{W_i}{\sqrt{2\pi} \cdot A_1} e^{-\frac{k_1^2}{2}} - \frac{W_i}{\sqrt{2\pi} \cdot A_1} e^{-\frac{k_2^2}{2}}$$

where $k_1 = \Phi^{-1}(A_0)$ or $\Phi^{-1}(1 - A_{m_i} - A_{m_i-1} - \cdots - A_1)$;

$k_2 = \Phi^{-1}(A_1 + A_0)$ or $\Phi^{-1}(1 - A_{m_i} - A_{m_i-1} - \cdots - A_2)$;

$k_{u_i} = \Phi^{-1}(A_{u_i} + A_{u_i-1})$ or $\Phi^{-1}(1 - A_{m_i} - A_{m_i-1} - \cdots - A_{u_i}), u_i = 2, 3, \cdots, m_i$;.

and $V_{u_i} = \frac{W_i}{\sqrt{2\pi} \cdot A_{u_i}} e^{-\frac{k_{u_i}^2}{2}} - \frac{W_i}{\sqrt{2\pi} \cdot A_{u_i}} e^{-\frac{k_{u_i+1}^2}{2}}$.

Therefore, we have

$$W_i = \frac{\sqrt{2\pi} \cdot V_1 \cdot A_1}{e^{-\frac{k_1^2}{2}} - e^{-\frac{k_2^2}{2}}} = \frac{\sqrt{2\pi} \cdot V_{u_i} \cdot A_{u_i}}{e^{-\frac{k_{u_i}^2}{2}} - e^{-\frac{k_{u_i+1}^2}{2}}}, u_i = 2, 3, \cdots, m_i.$$

Here, $k_1 = \Phi^{-1}(A_0)$ or $\Phi^{-1}(1 - A_{m_i} - A_{m_i-1} - \cdots - A_1)$;

$k_2 = \Phi^{-1}(A_1 + A_0)$ or $\Phi^{-1}(1 - A_{m_i} - A_{m_i-1} - \cdots - A_2)$;

$k_{u_i} = \Phi^{-1}(A_{u_i} + A_{u_i - 1})$ or $\Phi^{-1}(1 - A_{m_i} - A_{m_i - 1} - \cdots - A_{u_i}), u_i = 2, 3, \cdots, m_i$.

So, we complete the proof.

Note 2 By Theorem 4.2, we respectively obtain the formulas of $a_i, b_{u_i} (u_i = 1, \cdots, m_i)$ as follows

$$\begin{cases} b_1 = \dfrac{\Phi^{-1}(A_0)}{W_i} = \dfrac{\Phi^{-1}(1 - A_{m_i} - A_{m_i - 1} \cdots A_1)}{W_i} \\ b_2 = \dfrac{\Phi^{-1}(A_1 + A_0)}{W_i} = \dfrac{\Phi^{-1}(1 - A_{m_i} - A_{m_i - 1} \cdots A_2)}{W_i} \\ \cdots \\ b_{m_i} = \dfrac{\Phi^{-1}(A_{m_i - 1} + A_{m_i - 2})}{W_i} = \dfrac{\Phi^{-1}(1 - A_{m_i})}{W_i} \end{cases}$$

and $a_i = W_i / \sqrt{1 - W_i^2}$. By Theorem 4.3, we get

$$W_i = \frac{\sqrt{2\pi} \cdot V_1 \cdot A_1}{e^{-\frac{k_1^2}{2}} - e^{-\frac{k_2^2}{2}}} = \frac{\sqrt{2\pi} \cdot V_{u_i} \cdot A_{u_i}}{e^{-\frac{k_{u_i}^2}{2}} - e^{-\frac{k_{u_i+1}^2}{2}}}, u_i = 2, 3, \cdots, m_i.$$

In practice, if we approximately take $\hat{P}_{u_i}(\theta)$ as the estimation of A_{u_i} , and if we could find out the estimation of V_{u_i} through practical scores, then we have

$$\hat{W}_i = \frac{\sqrt{2\pi} \cdot \hat{V}_1 \cdot \hat{P}_1(\theta)}{e^{-\frac{k_1^2}{2}} - e^{-\frac{k_2^2}{2}}} = \frac{\sqrt{2\pi} \cdot \hat{V}_{u_i} \cdot \hat{P}_{u_i}(\theta)}{e^{-\frac{k_{u_i}^2}{2}} - e^{-\frac{k_{u_i+1}^2}{2}}}, u_i = 2, 3, \cdots, m_i.$$

So, we get $a_i = \hat{W}_i / \sqrt{1 - \hat{W}_i^2}$ and

$$\begin{cases} \hat{b}_1 = \dfrac{\Phi^{-1}(\hat{P}_0)}{\hat{W}_i} = \dfrac{\Phi^{-1}(1 - \hat{P}_{m_i} - \hat{P}_{m_i - 1} - \cdots - \hat{P}_1)}{\hat{W}_i} \\ \hat{b}_2 = \dfrac{\Phi^{-1}(\hat{P}_1 + \hat{P}_0)}{\hat{W}_i} = \dfrac{\Phi^{-1}(1 - \hat{P}_{m_i} - \hat{P}_{m_i - 1} - \cdots - \hat{P}_2)}{\hat{W}_i} \\ \cdots \\ \hat{b}_{m_i} = \dfrac{\Phi^{-1}(\hat{P}_{m_i - 1} + P_{m_i - 2})}{\hat{W}_i} = \dfrac{\Phi^{-1}(1 - \hat{P}_{m_i})}{\hat{W}_i} \end{cases}$$

VI. CONCLUSIONS

In this paper, we obtain estimations of parameters $a_i, b_{u_i} (u_i = 0, 1, \cdots, m_i)$ for graded response model. It is convenient and proper, when the sample size is small and the requirement for precision of estimation is not high. Well, the parameters we have obtained by this way can also be used as the initial value of joint maximum likelihood estimation, and it is helpful to avoid the occurrence of the problems discussed in Section 2.

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