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Common Fixed Points of Using Variants of Compatible Mappings

Sonia Rani
A.S. College for Women, Khanna, Punjab

Abstract: *In this paper, we generalize the notion of reciprocal continuity to conditionally reciprocal continuity and obtain related common fixed point theorem in metric and fuzzy metric spaces. We also demonstrate existence and uniqueness of common fixed point for a pair of conditionally compatible self-mappings.*

Keywords: *Conditionally reciprocal continuity, conditionally compatible mappings, g-compatible or f-compatible mappings*

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I. INTRODUCTION AND PRELIMINARIES

Zadeh [13] introduced the concept of fuzzy set. After the appearance of the notion of fuzzy set, the literature in area of fixed point theory has risen very quickly because of its interesting applications in applied sciences. Inspired by Menger spaces, Kramosil and Michalek [5] generalized the concept of probabilistic metric space to the fuzzy framework. Thereafter, many authors (see e.g.,[1-12]) established fuzzy version of most of the classical metric common fixed point theorems.

In the present paper, we generalize the notion of reciprocal continuity to conditionally reciprocal continuity and obtain related common fixed point theorem in metric and fuzzy metric spaces. We also demonstrate existence and uniqueness of common fixed point for a pair of conditionally compatible self-mappings.

Definition 1.1.[12] A binary operation $*$: $[0,1]\times[0,1] \rightarrow [0,1]$ is **continuous *t*-norm** if $*$ satisfies the following conditions: for all $a, b, c, d \in [0,1]$,

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$;
(iv) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$.

Definition 1.2.[5] The 3-tuple $(X, M, *)$ is said to be a **fuzzy metric space** if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

(i) $M(x, y, 0) = 0$;
(ii) $M(x, y, t) = 1$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$;
(v) $M(x, y, \bullet) : [0, \infty) \rightarrow [0,1]$ is left continuous;

The function $M(x, y, t)$ denote the degree of nearness between x and y w.r.t. t respectively.

Definition 1.3.[4] A pair (f, g) of self-mappings of a fuzzy metric space $(X, M, *)$ is said to be **compatible** if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

A pair (f, g) of self-mappings of a fuzzy metric space $(X, M, *)$ is said to be **non-compatible** if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \in X$ but $\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or non-existent for at least one $t > 0$.

Definition 1.4.[7,11] A pair (f, g) of self-mappings of a fuzzy metric space $(X, M, *)$ is said to be

(i) ***f*- compatible** if $\lim_{n\to\infty} M(fgx_n, ggx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

(ii) ***g*- compatible** if $\lim_{n\to\infty} M(gfx_n, ffx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

In 1998, Pant [8] introduced the concept of reciprocal continuity as follows:

Definition 1.5. Let *X* be any non-empty set. A pair (*f*, *g*) of self-mappings on set *X* is said to be **reciprocally continuous** if $\lim_{n\to\infty} fgx_n = fz$ and $\lim_{n\to\infty} gfx_n = gz$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

Definition 1.6.[1] Let *X* be any non-empty set. A pair (*f*, *g*) of self-mappings on set *X* is said to be **occasionally weakly compatible (*owc*)** if there exists a point $x \in X$ which is a coincidence point of *f* and *g* at which *f* and *g* commute.

Definition 1.7.[9] Let *X* be any non-empty set. A pair (*f*, *g*) of self-mappings on set *X* is said to be **conditionally reciprocally continuous** mappings if whenever the set of sequence $\{x_n\}$ satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ is nonempty, there exists a sequence $\{y_n\}$ in X satisfying $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = t$ such that $\lim_{n\to\infty} fgy_n = ft$ and $\lim_{n\to\infty} gfy_n = gt$.

Inspired by Pant and Bisht [10], we introduced similar concept in fuzzy metric spaces as follows:

Definition 1.8. A pair (*f*, *g*) of self-mappings of a fuzzy metric space $(X, M, *)$ is said to be **conditionally compatible** if whenever the set of sequence $\{x_n\}$ satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ is nonempty, there exists a sequence $\{y_n\}$ in *X* satisfying $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = t$, $\lim_{n\to\infty} M(fgy_n, gfy_n, t) = 1$.

Lemma 1.1.[6] Let $(X, M, *)$ be a fuzzy metric space. If there exists a constant $k \in (0,1)$ such that $M(x, y, kt) \geq M(x, y, t)$ for all $x, y \in X$, $t > 0$, then $x = y$.

II. MAIN RESULTS

Theorem 2.1: Let *f* and *g* be conditionally compatible self-mappings of a fuzzy metric space (*X*, *M*, *) satisfying:

(i) $M(x, gx, t) \neq \min\{M(x, fx, t), M(gx, fx, t)\}$, whenever the right hand side is $\neq 1$.

If *f* and *g* are noncompatible and reciprocally continuous, then *f* and *g* have a common fixed point.

Proof: Since *f* and *g* are noncompatible, there exists a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$ but either $\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or the limit does not exist. Also, as *f* and *g* are conditionally compatible and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, there exists a sequence $\{y_n\}$ in *X* satisfying $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = u$ such that $\lim_{n\to\infty} M(fgy_n, gfy_n, t) = 1$. Reciprocal continuity of *f* and *g* implies that $\lim_{n\to\infty} fgy_n = fu$ and $\lim_{n\to\infty} gfy_n = gu$. By using different limits, we get $fu = gu$. Lastly, we claim that *u* is a coincidence point of *g*. Suppose not, i.e., $u \neq gu$ then using (i), we have $M(u, gu, t) \neq \min\{M(u, fu, t), M(gu, fu, t)\}$, that is $M(u, gu, t) \neq M(u, gu, t)$, a contradiction. This gives, $u = gu = fu$. Hence *u* is a common fixed point of *f* and *g.*

Remark 2.1: Example 1.1 of [10] illustrates the validity of Theorem 2.1 if *I* and *T* in Example 1.1 are replaced by *f* and *g* respectively, over a standard fuzzy metric.

Theorem 2.2: Let f and g be conditionally reciprocal continuous self mappings of a complete fuzzy metric space $(X, M, *)$ such that:

(i) $fX \subseteq gX$;

(ii) $M(fx, fy, qt) \geq M(gx, gy, t), \; q \in (0,1)$.

If f and g are either compatible or g-compatible or f-compatible, then f and g have a unique common fixed point.

Proof: Let x_0 be any point in X. Then, since $fX \subseteq gX$, there exists a sequence of points $x_0, x_1, x_2, \ldots\ldots x_n, \ldots.$ such that $fx_n = gx_{n+1}$.

Define a sequence $\{a_n\}$ in X by

$$a_n = fx_n = gx_{n+1}, \quad n = 0,1,2,\ldots. \tag{2.1}$$

We claim that $\{a_n\}$ is a Cauchy sequence. Using (ii) we obtain

$$M(a_n, a_{n+1}, t) = M(fx_n, fx_{n+1}, t) \geq M\left(gx_n, gx_{n+1}, t/q\right)$$

$$= M\left(a_{n-1}, a_n, t/q\right) \geq \ldots \geq M\left(a_0, a_1, t/q^n\right).$$

Moreover, for every integer $p > 0$, we get

$$M(a_n, a_{n+p}, t) \geq M\left(a_n, a_{n+1}, t/p\right) * M\left(a_{n+1}, a_{n+2}, t/p\right) * \ldots M\left(a_{n+p-1}, a_{n+p}, t/p\right)$$

$$\geq M\left(a_0, a_1, t/pq^n\right) * M\left(a_0, a_1, t/pq^{n+1}\right) * \ldots M\left(a_0, a_1, t/pq^{n+p-1}\right).$$

This implies that $\lim_{n\to\infty} M(a_n, a_{n+p}, t) = 1$. Therefore $\{a_n\}$ is a Cauchy sequence. Since X is complete, there exists a point z in X such that $a_n \to z$ as $n \to \infty$. Moreover, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$.

Since f and g are conditionally reciprocally continuous and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, there exists a sequence $\{y_n\}$ satisfying $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = u(\text{say})$ such that $\lim_{n\to\infty} fgy_n = fu$ and $\lim_{n\to\infty} gfy_n = gu$. Since $fX \subseteq gX$, for each y_n, there exists a z_n in X such that $fy_n = gz_n$. Thus $\lim_{n\to\infty} gz_n = u$. By virtue of this and using (ii) we have $\lim_{n\to\infty} fz_n = u$. Therefore, we have

$$\lim_{n\to\infty} gy_n = u, \; \lim_{n\to\infty} fz_n = u, \; \lim_{n\to\infty} fy_n = \lim_{n\to\infty} gz_n = u. \tag{2.2}$$

Suppose that f and g are compatible. Then $\lim_{n\to\infty} M(fgy_n, gfy_n, t) = 1$ which implies $fu = gu$. Again, since compatibility implies commutativity at coincidence points, we get $gfu = fgu = ffu = ggu$. Using (ii) we get $M(fu, ffu, qt) \geq M(gu, ggu, t) = M(fu, ffu, t)$ which results, $fu = ffu$. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Now suppose that f and g are g-compatible. Then $\lim_{n\to\infty} M(ffy_n, gfy_n, t) = 1$ that is, $\lim_{n\to\infty} ffy_n = gu$. Using (ii) we obtain $M(fu, ffy_n, qt) \geq M(gu, gfy_n, t)$ on letting $n \to \infty$ we have $fu = gu$. Since g-compatibility implies commutativity at coincidence points, we obtain $fgu = gfu$ and $gfu = fgu = ffu = ggu$. Using (ii) we get $M(fu, ffu, qt) \geq M(gu, gfu, t) = M(fu, ffu, t)$, that is $fu = ffu$. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Finally suppose that f and g are f-compatible. Then $\lim_{n\to\infty} M(fgz_n, ggz_n, t) = 1$. By virtue of (2.2) and $\lim_{n\to\infty} gfy_n = \lim_{n\to\infty} ggz_n = gu$ we get $\lim_{n\to\infty} fgz_n = gu$. Using (ii) we get $M(fu, fgz_n, qt) \geq M(gu, ggz_n, t)$. On letting $n \to \infty$ this gives $fu = gu$. Since f-compatibility implies commutativity at coincidence points, we get $fgu = gfu$. Again using (ii), we have $M(fu, ffu, qt) \geq M(gu, gfu, t) = M(fu, ffu, t)$, that is $fu = ffu$. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Uniqueness of common fixed point theorem follows easily in each of the three cases.

Remark 2.2: Example 6 of [9] illustrates the validity of Theorem 2.2 over a standard fuzzy metric.

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