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# State Feedback Stabilization of a Single Delay Linear System: A Lyapunov-Krasovskii Approach

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***Abstract***
*The issue of state feedback stabilization of a Linear Time Invariant system (LTI) with a single constant delay in the states, is the subject of this paper. Although an independent delay stability is used to prove that the system can be stabilized, the considered system is actually delay-dependent stable. A maximum value of the delay $h_{max}$, beyond which stability is lost, is determined by trials and errors. The so called Lyapunov-Krasovskii functional approach is used. It leads to a Ricatti equation whose solution provides a memoryless stabilizing feedback. Simulation results show the effectiveness of the method.*
***Keywords:*** *Delay Dependent/Independent Stabilization, Lyapunov-Krasovskii functional, Ricatti equation, State feedback,Time Delay Systems.*

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## I. INTRODUCTION

The existence of delay in a system is a source of instability and makes system analysis and control design more complicated. Most often, it is a source of poor performances. In this study, a direct stabilizing state feedback is considered for an LTI system with a single constant delay. Simulated results are given and show that such a design guarantees stability of the closed loop system.

This paper is organized as follows. Introduction is Section I. The description of the system is in the first sub-section 1 of the introduction. Sub-section 2 is a brief literature review and background on Time Delays Systems (TDS). In section II, some mathematical preliminaries on matrices, relevant to this study, are recalled. Then, a fundamental result on delay independent stability is recalled and proved. The Lyapunov Krasovskii approach is brought about, to carry out a feedback stabilization in the system. In sub-section 4, an example is given and plots of the states show the effectiveness of the simulation results. Finally Section III is conclusion followed by the references in section IV.

### 1.1. System Description

In this section a description of the delayed system is given. A single input system with a single fixed state-delay is considered in (1). Vectors can be written in bold, if necessary to avoid confusion.

$$\dot{\boldsymbol{x}}(t) = A_0\boldsymbol{x}(t) + A_1\boldsymbol{x}(t-h) + Bu_f\boldsymbol{x}(t)(t); \quad t > 0 \;\; ; \;\; 0 \le h \le h_{1max} \qquad \dots (1)$$

$\boldsymbol{x}(t) = [x_1(t) \quad ... \quad x_n(t)]^T$ ; $\boldsymbol{x}(t) = \boldsymbol{\varphi}(t)$, for $t \,\epsilon[-h\,;\; 0]$ ; is known.

$\boldsymbol{\varphi}(t) : (n \times 1)$ is a known continuous vector valued initial state condition function;

$\boldsymbol{x}(t)$: $(n \times 1)$ is the measurable state vector ;

$u_f\boldsymbol{x}(t)$ is the scalar control input;

$\mathbb{R}$ is the set of real numbers; $V(\boldsymbol{x}, t)$ is a scalar positive definite Lyapunov-krasovskii functional ;

$B$ : $(n \times 1)$ ; is the known constant input matrix, with full rank ;

$h$ is the constant scalar, always positive, time- delay ;

$A_0$ , $A_1$ : $(n \times n)$ ; are the known constant system matrices .

The objective is to design a suitable *stabilizing state feedback* control law $u_f\boldsymbol{x}(t)$ for the system (1).

### 1.2. A Brief Review And Background On TDS

A well known feature of TDS is that, unlike systems without delay, the initial condition state vector $\boldsymbol{x}(t) = \boldsymbol{\varphi}(t)$, must be known on an entire intervalle: $t \,\epsilon\, [-h\,;\; 0]$, and not only $x_0 = \boldsymbol{x}(t_0)$ at a single value of $t = t_0$.

There is another feature. As it has been well reported in [1], the time delay system described in (1) is said to be infinite dimensional, because its characteristic polynomial in (2), has an infinite number of roots [1].

$$P(s) = \det(sI - A_0 - A_1 e^{-sh}) = 0 \qquad \ldots \qquad (2)$$

The infinite dimensional nature of the system (1) requires a new approach as in [2] , [3] and [4], when using the Lyapunov theory. In applying the so called Lyapunov- Krasovskii approach, one is reminded that, it is in [5] that was suggested for the first time the use of a functional instead of an ordinary classical Lyapunov function. Most stability analysis such as in [6] and [7], is based on that approach. Current trends in Variable Structure Controller (VSC) design as in [8], [9], mainly refer to this approach to constructing a sliding surface. A design relying on a classical Lyapunov function, was carried out in [10], for a system without delay.

In this paper, we show that the result given in [11] and [12], is consistent with the Lyapunov Krasovskii approach.

## II. LYAPUNOV-KRASOVSSKII APPROACH

### 2.1. Mathematical Preliminaries

For any matrix $P$ , $P^T$ is the transpose of $P$ . If $P^T = P$, then $P$ is symmetric.

If a matrix $P$ is symmetric, the $n$ eigenvalues are real. The eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal. For a **symmetric positive definite matrix,** $\boldsymbol{P^T = P > O}$ meaning that all eigenvalues are positive.

#### 2.1.1. Equivalent Statements For A Definite Matrix

Given two positive definite matrices $M$ , $N$ ; it can be shown that $M + N$ is also positive definite. If $M > O$ (positive definite) , then $-M < O$ (negative definite).

For any given positive definite square matrix $P > O$ of size $n$, the following statements (i), … (v ) from [13], are equivalent.

(i). The $n$ pivots of $P$ are strictly positive (they are reals).

(ii). The $n$ determinants of the main diagonal of the matrix $P$ are positive.

(iii). The $n$ eigenvalues of $P$ are strictly positive (they are reals).

(iv). For any vector $x = x(t) \in \mathbb{R}^n$ , if $x(t) \neq 0$ ; we have $x^T P\, x > 0$ . This definition, based on the energy (Lyapunov's function) is fundamental in control systems.

(v). $P = R^T R$ where $R$ has its columns linearly independent.

(vi). The Cholesky's decomposition $P = LL^T$ is possible.

The $n$ determinants referred to in (ii), above, are defined as follow.

$$|P_1| = [p_{11}];\ |P_2| = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix};\ldots;\ |P_n| = \det(P).$$

### 2.2. Delay Independent Lyapunov-Krasovsskii Approach

Consider the system (1) without the input control $u(t)$ and the Lyapunov-Krasovskii functional $V(\boldsymbol{x}, t)$ in (3). This functional satisfies (i) and (ii)) and is said to be positive definite.

(i). $V(\boldsymbol{x}, t) = 0$, if and only if $\boldsymbol{x} = 0$.

(ii). $V(x, t) \geq 0$ for all $x\,(t) \in \mathbb{R}^n$, and for all $t \in \mathbb{R}$.

$$V(\,x, t) = x^T(t)Px(t) + \int_{-h}^{0} x^T(t+\theta)Qx(t+\theta)d\theta \qquad \ldots \qquad (3)$$

Where $P$ and $Q$ are symmetric positive definite: $P = P^T > O$ and $Q = Q^T > O$.

$V(x, t)$ is decreasing if its time derivatve $\frac{dV}{dt} = \dot{V}(x(t))$ is negative. According to Lyapunov stability theory (that the reader is assumed to be familiar with) the system is stable if $V(x, t)$ is decreasing along the trajectories of the system. The existence of a Lyapunov function is a sufficient condition only (not necessary) for system stability.

If $\frac{dV}{dt} = \dot{V}(x(t)) < 0$ along the trajectories, then the system is stable.

$\frac{dV}{dt} = \dot{V}(x(t)) = \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) + x^T(t)Qx(t) - x^T(t-h)Qx^T(t-h) =$
$[A_0x(t) + A_1x\left(t-h\right)]^T Px(t) + x^T(t)P\left[A_0x(t) + A_1x\left(t-h\right)\right] + x^T(t)Qx(t) - x^T(t-h)Qx^T(t-$
$h$= [$xTtA0T$+$xTt$−$hA1T$] $Pxt$+$xTtPA0xt$+ $A1x$ $t$ −$h$+$xTtQxt$−$xTt$−$hQxTt$−$h$= [$xTtA0TPxt$+$xTt$−$hA1T$
$Px(t)] + x^T(t)PA_0x(t) + x^T(t)PA_1x\left(t-h\right) + x^T(t)Qx(t) - x^T(t-h)Qx^T(t-h)$

$$= \begin{pmatrix} x(t) \\ x(t-h) \end{pmatrix}^T \begin{pmatrix} PA_0 + {A_0}^T P + Q & PA_1 \\ A_1^T P & -Q \end{pmatrix} \begin{pmatrix} x(t) \\ x(t-h) \end{pmatrix} < O$$

The next result (4) is reported in [3],, to have been proven for the first time in [13]. The system is asymptotically stable for any delay if there exist positive symmetric matrices $P = P^T > O$ and $Q = Q^T > O$, such that (4) holds.

$$\begin{pmatrix} PA_0 + {A_0}^T P + Q & PA_1 \\ A_1^T P & -Q \end{pmatrix} < O \quad \ldots (4)$$

In the literature, this result is referred to as a *delay-independent* stability criterion, because the delay does not explicitly appear in (4).

### 2.3. Feedback Stabilization

**Theorem**

The autonomous system in (5) can be stabilized by the feedback control law (6) as in the controlled system (7). The closed loop system in (8) is asymptotically stable.

$$\dot{\boldsymbol{x}}(t) = A_0 \boldsymbol{x}(t) + A_1 \boldsymbol{x}(t-h) \quad \ldots (5)$$

$$u_f(t) = -B^T P x(t) \quad \ldots (6)$$

$$\dot{\boldsymbol{x}}(t) = A_0 \boldsymbol{x}(t) + A_1 \boldsymbol{x}(t-h) + B u_f(t) \quad \ldots (7)$$

$$\dot{x} = (A_0 - BB^T P)x(t) + A_1 x(t-h) \quad \ldots (8)$$

Where $P = P^T \geq O$ is solution to the Riccati equation in (9)

$${A_0}^T P + PA_0 - PBB^T P + Q = 0 \quad \ldots (9)$$

**Proof**

It is easy to show (8).

Plugging the control (7) into the system (5) gives the expected result.

So $\dot{\boldsymbol{x}}(t) = A_0 \boldsymbol{x}(t) + A_1 \boldsymbol{x}(t-h) + B u_f \boldsymbol{x}(t)$

Where $u_f(t) = -B^T P x(t)$

$$\dot{\boldsymbol{x}}(t) = A_0 \boldsymbol{x}(t) + A_1 \boldsymbol{x}(t-h) + B u_f(t) = \dot{\boldsymbol{x}}(t) = A_0 \boldsymbol{x}(t) + A_1 \boldsymbol{x}(t-h) + B(t - B^T P \boldsymbol{x}(t))$$
$$= (A_0 - BB^T P)\boldsymbol{x}(t) + A_1 \boldsymbol{x}(t-h).$$

To derive (9), we choose the Lyapunov-Krasovskii functional candidate as $V(x,t) \geq 0$ in (9).

$$V(\boldsymbol{x},t) = x^T(t) P x(t) + k \int_{t-h}^{t} x^T(\theta) Q_1 x(\theta) d\theta \quad \ldots (9)$$

$$\dot{V}(\boldsymbol{x},t) = \frac{dV}{dt} = x^T(t)\left(PA_0 + {A_0}^T P\right)x(t) + 2x^T(t) PA_1 x(t-h) + k[x^T(t) Q_1 x(t) - x^T(t-h) Q_1 x(t-h)] - 2k(PBB^T P)x(t) = \begin{pmatrix} x(t) \\ x(t-h) \end{pmatrix}^T \begin{pmatrix} PA_0 + {A_0}^T P - 2k(PBB^T P) + kQ_1 & PA_1 \\ A_1^T P & -kQ_1 \end{pmatrix} \begin{pmatrix} x(t) \\ x(t-h) \end{pmatrix} \leq O .$$

Therefore $$\begin{pmatrix} PA_0 + {A_0}^T P - 2k(PBB^T P) + kQ_1 & PA_1 \\ A_1^T P & -kQ_1 \end{pmatrix} \leq O$$

Which implies $PA_0 + {A_0}^T P - 2k(PBB^T P) + kQ_1 \leq O$

There exists $R_1 \geq O$ so that

$$PA_0 + {A_0}^T P - 2k(PBB^T P) + kQ_1 + R_1 = O$$

Setting $Q = kQ_1 + R_1 \geq O$ and choosing $k = 1/2$.

Finally $PA_0 + A_0{}^T P - (PBB^T P) + Q = O$

**2.4. Implementation And Result**

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + B\,u(x,t)$$
$$x(t) = \phi(t) \quad if \quad t \,\epsilon [-h;\ 0] \ , \ h > 0\,. \qquad \ldots \qquad (10)$$

$$x(t) = [x_1 \quad x_2 \quad x_3]^{\,T};\ \ t > 0.$$
$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{bmatrix};\ A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\beta_0 & -\beta_1 & -\beta_2 \end{bmatrix};\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $Q = 2(\beta_0^2 + \beta_1^2 + \beta_2^2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\alpha_0 = 0.3240\ ; \quad \alpha_1 = 1.8000; \quad \alpha_2 = 2.7000; \qquad \ldots \qquad (11)$$

$$\beta_0 = \alpha_0/2\ ; \quad ;\ \beta_1 = \alpha_1/2 \quad \beta_2 = \alpha_2/2;$$
$$x(t) = \ \phi(t) = [10 \quad 0 \quad -5]^{\,T} \ if \quad t \,\epsilon [-h;\ 0]$$

$$\dot{x}_1 = \quad x_2$$
$$\dot{x}_2 = \quad x_3$$
$$\dot{x}_3 = \quad -\alpha_0\, x_1(t) - \alpha_1\, x_2(t) - \alpha_3 x_2(t) - \beta_0 x_1(t-h) - \beta_1 x_2(t-h) - \beta_2 x_3(t-h) + u(x,t)$$

$$= \begin{pmatrix} 11.0921 & 8.7530 & 2.1497 \\ 8.7530 & 15.5030 & 3.9614 \\ 2.1497 & 3.9614 & 2.5311 \end{pmatrix}. \qquad B^T P = [2.1497 \quad 3.9614 \quad 2.5311]$$

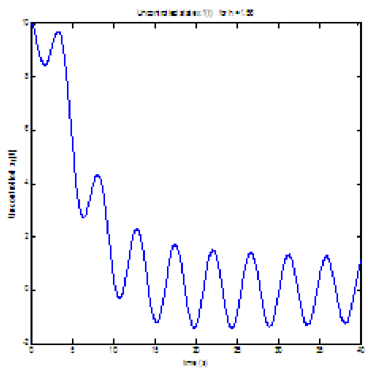

Fig.1- Unstabilized state x_1 (t) for h=1.56

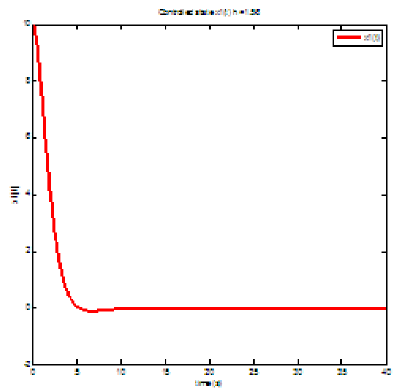

Fig.2- Stabilized state $x_1(t)$ for $h = 1.56$

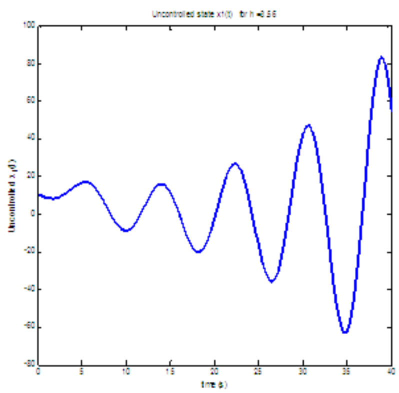

Fig.3- Unstabilized state $x_1(t)$ for $h = 3.56$

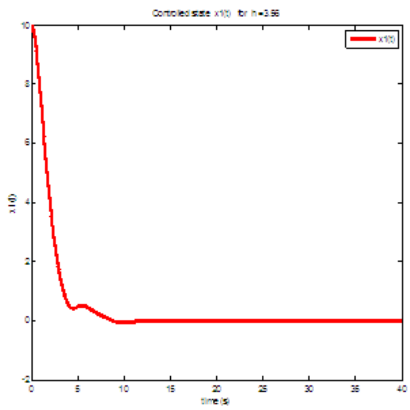

Fig.4- Stabilized state $x_1(t)$ for $h = 3.56$

## III. CONCLUSION

Stabilization by a simple state feedback of a linear TDS with a single constant delay in the states has been investigated. The Lyapunov-Krasovskii functional based approach is at the heart of the process. It leads to a Riccati equation. Simulation results show the effectiveness of the method.

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