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Existence and stability of positive solutions for a nonlinear delay difference equation

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Abstract
In this paper, we study the existence and stability of positive solutions for a nonlinear delay difference equation. The main tools employed here are the Schauder's fixed point theorem and the Lyapunov function method. The main results are illustrated with two examples.
Keywords: *Difference equation; Positive solution; Stability; Delay.*

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I. INTRODUCTION

In the past two decades, difference equation plays an important role in application; see [1-8]. In [9], the author studied the existence of positive periodic solutions of a nonlinear delay difference equation

$$\Delta x(n)+\sum_{s=n-\tau}^{n-1} p(n,s)g(x(s))=0, n>T, \tag{1.1}$$

where $x:\mathbb{Z}\to\mathbb{R},\Delta$ denotes the forward difference operator, $\Delta x(n)=x(n+1)-x(n)$,

$p\in C(\mathbb{Z}\times\mathbb{Z},\mathbb{R}), g\in C((0,\infty),(0,\infty)),\tau,T\in\mathbb{Z}.$

Based on the work in [9], the main purpose of this paper is to study the existence and stability of positive solutions of the equation (1.1). The existence results of the equation (1.1) will be obtained by the Schauder's fixed point theorem.

Theorem 1.1 (Schauder's fixed point theorem [10]). Let Ω be a closed, convex and nonempty subset of a Banach space X. Let $S:\Omega\to\Omega$ be a continuous mapping such that $S\Omega$ is a relatively compact subset of X. Then S has at least one fixed point in Ω. That is there exists an $x\in\Omega$ such that $Sx=x$.

II. EXISTENCE OF POSITIVE SOLUTIONS

In this section we shall study the existence of positive solutions of (1.1).

Theorem 2.1 Suppose that there exists a positive continuous function $k(n,s), n-\tau\le s\le n$, such that

$$\begin{aligned}&\exp\Big(\sum_{u=T}^{n}\ln\Big(1+\sum_{v=u-\tau}^{u-1} p(u,v)k(u,v)\Big)\Big)\\ &\times g\Big(\exp\Big(-\sum_{u=T}^{s-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1} p(u,v)k(u,v)\Big)\Big)\Big)=k(n,s), n>T,\end{aligned} \tag{2.1}$$

and

$$\sum_{s=n-\tau}^{n-1} p(n,s)k(n,s)>0, n>T. \tag{2.2}$$

Then (1.1) has a positive solution

$$x(n)=\exp\Big(-\sum_{u=T}^{n-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)k(u,v)\Big)\Big), n>T.$$

Proof: Let $X_1=x\in C([T-\tau,\infty),\mathbb{R})$ be the set of all continuous bounded functions. Then X_1 is a Banach space with the norm $\|x\|=\sup_{n\geq T-\tau}|x(n)|$. We set

$$w(n)=\exp\Big(-\sum_{u=T}^{n}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)k(u,v)\Big)\Big), n>T.$$

We define a closed, bounded and convex subset Ω_1 of X_1 as follows

$$\Omega_1=\{x\in X_1: w(n)\leq x(n)\leq 1, n>T,$$
$$k(n,s)x(n+1)=g(x(s)), n>T, n-\tau\leq s\leq n, x(n)=1, T-\tau\leq n\leq T\}.$$

Define the operator $S_1:\Omega_1\to X_1$ as follows

$$(S_1x)(n)=\begin{cases}\exp\Big(-\sum_{u=T}^{n-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)\dfrac{g(x(v))}{x(u+1)}\Big)\Big), n>T,\\ 1, T-\tau\leq n\leq T.\end{cases}$$

For every $x\in\Omega_1$ and $n>T$ we obtain

$$(S_1x)(n)=\exp\Big(-\sum_{u=T}^{n-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)k(u,v)\Big)\Big)\leq 1,$$

and $(S_1x)(n)\geq w(n)$.

For $n\in[T-\tau,T]$, we get $(S_1x)(n)=1$, that is $(S_1x)(n)\in\Omega$.

Now we can proceed by the similar way as in the proof of Theorem 2.1 in [9], by using the Schauder's fixed point theorem, we can obtain the results. We omit the rest of the proof.

Corollary 2.1 Assume that there exists a positive and continuous function $k(n,s), n-\tau\leq s\leq n$, such that

$$\exp\Big(\sum_{u=s}^{n}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)k(u,v)\Big)\Big)=k(n,s), n>T. \qquad (2.3)$$

and (2.2) hold. Then

$$\Delta x(n)+\sum_{s=n-\tau}^{n-1}p(n,s)x(s)=0, n>T, \qquad (2.4)$$

has a positive solution

$$x(n)=\exp\Big(-\sum_{u=T}^{n-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)k(u,v)\Big)\Big), n>T.$$

Corollary 2.2 Assume that there exists a positive and continuous function $k(n,s), n-\tau\leq s\leq n$, such that (2.1) and (2.2) hold and

$$\lim_{n\to\infty}\sum_{u=T}^{n-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)k(u,v)\Big)=\infty. \qquad (2.5)$$

Then (1.1) has a positive solution which tends to zero.

Corollary 2.3 Assume that there exists a positive and continuous function $k(n,s), n-\tau\leq s\leq n$, such that (2.1) and (2.2) hold and

$$\sum_{u=T}^{n-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u,v)k(u,v)\Big)=a. \tag{2.6}$$

Then (1.1) has a positive solution which tends to constant e^{-a}.

III. EXPONENTIAL STABILITY

In this section, we shall study the exponential stability of positive solution of (1.1). Assume that $x(n)$, $x_1(n)$ are two solutions of (1.1) , let $y(n)=x(n)-x_1(n),$ then

$$\Delta y(n)+\sum_{s=n-\tau}^{n-1}p(n,s)[g(x(s))-g(x_1(s))]=0, n>T,$$

By the mean value theorem we obtain

$$\begin{aligned}&\Delta y(n)+\sum_{s=n-\tau}^{n-1}p(n,s)g'(x_*)[x(s)-x_1(s)]=0, n>T,\\ &\Delta y(n)+\sum_{s=n-\tau}^{n-1}p(n,s)g'(x_*)y(s)=0, n>T.\end{aligned} \tag{3.1}$$

Definition 3.1 Let x_1 be a positive solution of (1.1) and there exist constants T_{x_1}, K_{x_1} and $\lambda>0$ such that for every solution x of (1.1)

$$|x(n)-x_1(n)|\le K_\varphi\left(\frac{1}{1+\lambda}\right)^n, n\ge T_{x_1}.$$

Then x_1 is said to be exponentially stable.

Theorem 3.1 Suppose that (2.1) and (2.2) hold and

$$p\in C([T,+\infty)_{\mathbb{T}}\times[0,\tau]_{\mathbb{T}},(0,+\infty]), g\in C^1((0,+\infty),(0,+\infty)), g'(x)\ge c>0.$$

Then (1.1) has a positive solution which is exponentially stable.

Proof. We shall show that there exists a positive λ such that

$$|x(n)-x_1(n)|\le K_{x_1}\left(\frac{1}{1+\lambda}\right)^n, n\in[T_1,+\infty)_{\mathbb{T}}, T_1\ge T-\tau,$$

where $K_{x_1}=\max_{n\in[T-\tau,T_1]}|y(n)|(1+\lambda)^{T_1}+1$.

Consider the following Lyapunov function

$$V(n)=|y(n)|(1+\lambda)^n, n\in[T_1,+\infty).$$

We claim that $V(n)\le K_{x1}$ for $n\in[T_1,+\infty)$.

On the other hand, there exists $n_*\in[T_1,+\infty)$, and n_* is the first constant such that $V(n_*)=K_{x1}$ or $V(n_*)>K_{x1}$.

Calculating the difference of $V(n)$ along the solution y of (3.1) we obtain

$$\Delta V(n)=(1+\lambda)^n(|y(n+1)|-|y(n)|)+\lambda(1+\lambda)^n|y(n+1)|.$$

For $n=n_*$, we get

$$0\le\Delta V(n_*)=(1+\lambda)^{n_*}(-c\sum_{s=n-\tau}^{n-1}p(n,s)|y(n_*)|)+\lambda(1+\lambda)^{n_*}|y(n_*+1)|.$$

If $y(n)>0, n\in[T,+\infty)$ then from (3.1) it follows that for $n\in[T_1,+\infty)$ the function y is decreasing and if $y(n)<0, n\in[T,+\infty)$ then y is increasing for $n\in[T_1,+\infty)$. We conclude that

$|y(n)|, n\in[T_1,+\infty)$ has decreasing character. Then we obtain

$$0\le \Delta V(n_*)=(1+\lambda)^{n_*}(-c\sum_{s=n-\tau}^{n-1}p(n,s)|y(n_*)|)+\lambda(1+\lambda)^{n_*}|y(n_*+1)|$$

$$\le(1+\lambda)^{n_*}|y(n_*)|[-c\sum_{s=n_*-\tau}^{n_*-1}p(n,s)+\lambda].$$

For $0<\lambda<c\sum_{s=n_*-\tau}^{n_*-1}p(n,s)$ we have a contradiction. Thus $|y(n)|(1+\lambda)^n\le K_{x1}$ for $n\in[T_1,+\infty)$ and $0<\lambda<c\sum_{s=n_*-\tau}^{n_*-1}p(n,s)$. This completes the proof.

IV. EXAMPLES

In this section, we give two examples to illustrate our main results.

Example 1. Let $p(n-s)=\frac{e-1}{\tau}e^{-(n-s)-1}$ and $g(x(s))=x(s)$, we choose

$$k(n,s)=e^{n-s+1}, n-\tau\le s\le n.$$

For the condition (2.1), we have

$$\exp\Big(\sum_{u=s}^{n}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u-v)k(u,v)\Big)\Big)=\exp\Big(\sum_{u=s}^{n}1\Big)=e^{n-s+1}=k(n,s), n\in[0,+\infty).$$

For the condition (2.5), we have

$$\lim_{n\to\infty}\sum_{u=T}^{n-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u-v)k(u,v)\Big)=\lim_{n\to+\infty}(n-T)=+\infty.$$

Therefore, all conditions of Corollary 2.2 are satisfied, then equation (1.1) has a positive solution

$$x(n)=\exp\Big(-\sum_{u=T}^{n-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u-v)k(u,v)\Big)\Big)=e^{-n+T}, n\in[0,+\infty),$$

which tends to zero.

Example 2. Let $p(n-s)=\frac{1}{e^2}\cos\pi(n-s), \tau=2$, and $g(x(s))=x(s)$, we choose

$$k(n,s)=e^{n-s+1}, n-\tau\le s\le n.$$

For the condition (2.1), we have

$$\exp\Big(\sum_{u=s}^{n}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u-v)k(u,v)\Big)\Big)=\exp\Big(\sum_{u=s}^{n}1\Big)=e^{n-s+1}=k(n,s), n\in[0,+\infty).$$

For the condition (2.5), we have

$$\lim_{n\to\infty}\sum_{u=T}^{n-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u-v)k(u,v)\Big)=\lim_{n\to+\infty}(n-T)=+\infty.$$

Therefore, all conditions of Corollary 2.2 are satisfied, then equation (1.1) has a positive solution

$$x(n)=\exp\Big(-\sum_{u=T}^{n-1}\ln\Big(1+\sum_{v=u-\tau}^{u-1}p(u-v)k(u,v)\Big)\Big)=e^{-n+T}, n\in[0,+\infty),$$

which tends to zero.

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