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Proof of a multivariate symmetric inequality by local majorization method

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Abstract
In this paper, we establish a multivariate symmetric inequality of four variables by using local majorization method and Schur convex.

Keywords: *Fractional inequality, Inequality conjecture, Local majorization, Schur convex function.*

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I. INTRODUCTION

In engineering application, statistical analysis and inequality proof, the extreme values of the following function are often involved.

$$F(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} f(x_i), \tag{1.1}$$

where $f(x)$ is usually a derivable function in interval $I \subset R$. Suppose that $f(x)$ is a second derivable function with at least one inflection point in interval I. Then it is not suitable to directly use Jensen's inequality to obtain its extreme value of (1.1) for I^n on hyperplane $x_1 + x_2 + \cdots + x_n = 1$. For example, Kang [2] prove the following inequality by using the properties of Jensen's convex function.

Proposition A. Suppose that $4 \le n \in N$, $x_1, x_2, \cdots, x_n$ are positive numbers. Then

$$\frac{1}{\sqrt{\frac{n+1}{x_1} - n}} + \frac{1}{\sqrt{\frac{n+1}{x_2} - n}} + \cdots + \frac{1}{\sqrt{\frac{n+1}{x_n} - n}} \ge 1. \tag{1.2}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n = \frac{1}{n}$.

But, Huang [3] pointed out that the Kang's proof for inequality (1.2) is mistake, and then gave the results contrary to inequality (1.2). However, we gave a counterexample to point out that the proofs of the two opposite conclusions in [2] and [3] are wrong, and proved the following inequality (see [4]).

Theorem B Suppose that $x_1, x_2, x_3 > 0$ $x_1 + x_2 + x_3 = 1$. Then

$$\frac{2}{\sqrt{5}} \le \sqrt{\frac{x_1}{4-3x_1}} + \sqrt{\frac{x_2}{4-3x_2}} + \sqrt{\frac{x_3}{4-3x_3}} \le \frac{\sqrt{3+8\sqrt{6}}}{3\sqrt{3}} + \frac{2\sqrt{4\sqrt{6}-9}}{3\sqrt{15}}. \tag{1.3}$$

In this paper, by using local majorization and Schur's convex function technical method, we will prove the following inequality.

Theorem 1 Suppose that $x_1, x_2, x_3, x_4 > 0$ $x_1 + x_2 + x_3 + x_4 = 1$. Then

$$\frac{\sqrt{6}}{3} \le \sqrt{\frac{x_1}{5-4x_1}} + \sqrt{\frac{x_2}{5-4x_2}} + \sqrt{\frac{x_3}{5-4x_3}} + \sqrt{\frac{x_4}{5-4x_4}} \le 1.065646364\ldots \quad (1.4)$$

II. THE PROOF OF THEOREM

1 Proof of Theorem 1.

Let $f(x) = \sqrt{\frac{x}{5-4x}}$, then $F := F(x_1, x_2, x_3, x_4) = \sum_{i=1}^{4} f(x_i)$. Write $a(x) = \sqrt{x(5-4x)^3}$, we have

$$f'(x) = \frac{5}{2} x^{-1/2}(5-4x)^{-3/2} = \frac{5}{2a(x)}. \quad (2.1)$$

$$f''(x) = 20x^{-3/2}(5-4x)^{-5/2}\left(x - \frac{5}{16}\right). \quad (2.2)$$

Thus $f(x)$ has a unique inflection point $x = 5/16$ in the interval $(0,\ 1)$.

Noticing that $F(x_1, x_2, x_3, x_4)$ is a symmetric function in the region: $0 < x_1, x_2, x_3, x_4 < 1$ and $x_1 + x_2 + x_3 + x_4 = 1$, we may assume that $1 > x_1 \ge x_2 \ge x_3 \ge x_4 > 0$. According to the different positions of the inflection point $x = 5/16$, it can be divided into the following four situations.

Case 1. If $5/16 \ge x_1 \ge x_2 \ge x_3 \ge x_4 > 0$, then $F(x_1, x_2, x_3, x_4)$ is a Schur concave function in the region $(0,\ 5/16]^4$. Since

$$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \prec (x_1, x_2, x_3, x_4) \prec \left(\frac{5}{16}, \frac{5}{16}, \frac{5}{16}, \frac{1}{16}\right), \quad (2.3)$$

therefore

$$3f\left(\frac{5}{16}\right) + f\left(\frac{1}{16}\right) \le F \le 4f\left(\frac{1}{4}\right). \quad (2.4)$$

Case 2. If $1 > x_1 \ge 5/16 \ge x_2 \ge x_3 \ge x_4 > 0$, then $F_1(x_2, x_3, x_4) := F(1 - \sum_{i=2}^{4} x_i, x_2, x_3, x_4)$ is a Schur concave function in the region $(0,\ 5/16]^3$. Since

$$\left(\frac{1-x_1}{3}, \frac{1-x_1}{3}, \frac{1-x_1}{3}\right) \prec (x_2, x_3, x_4) \prec \begin{cases} (1-x_1, 0, 0), & 0 < 1 - x_1 \le \frac{5}{16} \\ \left(\frac{5}{16}, \frac{11}{16} - x_1, 0\right), & \frac{5}{16} \le 1 - x_1 < \frac{10}{16} \\ \left(\frac{5}{16}, \frac{5}{16}, \frac{6}{16} - x_1\right), & \frac{10}{16} \le 1 - x_1 < \frac{11}{16} \end{cases}, \quad (2.5)$$

therefore

$$\left.\begin{array}{ll} f(x_1) + f(1-x_1), & \frac{11}{16} < x_1 < 1 \\ f(x_1) + f\left(\frac{11}{16} - x_1\right) + f\left(\frac{5}{16}\right), & \frac{6}{16} < x_1 \le \frac{11}{16} \\ f(x_1) + f\left(\frac{6}{16} - x_1\right) + 2f\left(\frac{5}{16}\right), & \frac{5}{16} \le x_1 \le \frac{6}{16} \end{array}\right\} \le F \le f(x_1) + 3f\left(\frac{1-x_1}{3}\right). \quad (2.6)$$

Case 3. If $1 > x_1 \ge x_2 \ge 5/16 \ge x_3 \ge x_4 > 0$, then $f(x_1)+f(x_2)$ is a Schur convex function in the region $[5/16,\ 1]^2$, and $f(x_3)+f(x_4)$ is a Schur concave function in the region $[0,\ 5/16]^2$.

(i) If $\frac{10}{16} \le x_1 + x_2 := t < \frac{11}{16}$, then $\frac{5}{16} < x_3 + x_4 = 1 - t \le \frac{6}{16}$. In this case, it holds the following majorization relations.

$$\left(\frac{x_1+x_2}{2},\frac{x_1+x_2}{2}\right) \prec (x_1,x_2) \prec \left(x_1+x_2-\frac{5}{16},\frac{5}{16}\right), \tag{2.7}$$

$$\left(\frac{x_3+x_4}{2},\frac{x_3+x_4}{2}\right) \prec (x_3,x_4) \prec \left(\frac{5}{16},x_3+x_4-\frac{5}{16}\right). \tag{2.8}$$

We have

$$2f\left(\frac{x_1+x_2}{2}\right) \le f(x_1)+f(x_2) \le f\left(x_1+x_2-\frac{5}{16}\right)+f\left(\frac{5}{16}\right), \tag{2.9}$$

$$f\left(x_3+x_4-\frac{5}{16}\right)+f\left(\frac{5}{16}\right) \le f(x_3)+f(x_4) \le 2f\left(\frac{x_3+x_4}{2}\right). \tag{2.10}$$

Thus

$$2f\left(\frac{t}{2}\right)+f\left(\frac{11}{16}-t\right)+f\left(\frac{5}{16}\right) \le F \le 2f\left(\frac{1-t}{2}\right)+f\left(t-\frac{5}{16}\right)+f\left(\frac{5}{16}\right),\ \frac{10}{16} \le t < \frac{11}{16}. \tag{2.11}$$

(ii) If $\frac{11}{16} \le x_1 + x_2 = t < 1$, then $0 < x_3 + x_4 = 1 - t \le \frac{5}{16}$. In this case, it holds the following majorization relations.

$$\left(\frac{x_1+x_2}{2},\frac{x_1+x_2}{2}\right) \prec (x_1,x_2) \prec \left(x_1+x_2-\frac{5}{16},\frac{5}{16}\right), \tag{2.12}$$

$$\left(\frac{x_3+x_4}{2},\frac{x_3+x_4}{2}\right) \prec (x_3,x_4) \prec \left(x_3+x_4,0\right). \tag{2.13}$$

We have

$$2f\left(\frac{x_1+x_2}{2}\right) \le f(x_1)+f(x_2) \le f\left(x_1+x_2-\frac{5}{16}\right)+f\left(\frac{5}{16}\right), \tag{2.14}$$

$$f(x_3+x_4) \le f(x_3)+f(x_4) \le 2f\left(\frac{x_3+x_4}{2}\right). \tag{2.15}$$

Thus

$$2f\left(\frac{t}{2}\right)+f(1-t) \le F \le 2f\left(\frac{1-t}{2}\right)+f\left(t-\frac{5}{16}\right)+f\left(\frac{5}{16}\right),\ \frac{11}{16} \le t < 1. \tag{2.16}$$

Case 4. If $1 > x_1 \ge x_2 \ge x_3 \ge 5/16 > x_4 > 0$, then $F_2(x_1,x_2,x_3) := F(x_1,x_2,x_3,1-\sum_{i=1}^{3} x_i)$ is a Schur convex function in the region $(5/16,\ 1]^3$. Since

$$\left(\frac{1-x_4}{3},\frac{1-x_4}{3},\frac{1-x_4}{3}\right) \prec (x_1,x_2,x_3) \prec \left(\frac{6}{16}-x_4,\frac{5}{16},\frac{5}{16}\right), \tag{2.17}$$

therefore

$$3f\left(\frac{1-x_4}{3}\right)+f\left(x_4\right) \le F \le f(x_4)+f\left(\frac{6}{16}-x_4\right)+2f\left(\frac{5}{16}\right). \tag{2.18}$$

Combining (2.6), (2.11), (2.16) and (2.18), we get

$$\left.\begin{array}{c} f(x)+f(1-x),\quad \frac{11}{16}<x<1 \\ f(x)+f\left(\frac{11}{16}-x\right)+f\left(\frac{5}{16}\right),\frac{6}{16}<x\le\frac{11}{16} \\ f(x)+f\left(\frac{6}{16}-x\right)+2f\left(\frac{5}{16}\right),\frac{5}{16}<x\le\frac{6}{16} \\ 2f\left(\frac{x}{2}\right)+f\left(\frac{11}{16}-x\right)+f\left(\frac{5}{16}\right),\frac{10}{16}\le x\le\frac{11}{16} \\ 2f\left(\frac{x}{2}\right)+f(1-x),\frac{11}{16}\le x<1 \\ f(x)+3f\left(\frac{1-x}{3}\right),0<x\le\frac{5}{16} \end{array}\right\}\le F\le\left\{\begin{array}{c} f(x)+3f\left(\frac{1-x}{3}\right),\frac{5}{16}\le x<1 \\ f\left(x-\frac{5}{16}\right)+2f\left(\frac{1-x}{2}\right)+f\left(\frac{5}{16}\right),\frac{10}{16}\le x<1 \\ f(x)+f\left(\frac{6}{16}-x\right)+2f\left(\frac{5}{16}\right),0<x<\frac{5}{16} \end{array}\right. . \tag{2.19}$$

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$$h_1(x):=f(x)+f(1-x),\quad h_2(x):=f(x)+f\left(\frac{11}{16}-x\right)+f\left(\frac{5}{16}\right),$$
$$h_3(x):=2f\left(\frac{x}{2}\right)+f(1-x),\quad h_4(x):=f(x)+f\left(\frac{6}{16}-x\right)+2f\left(\frac{5}{16}\right),$$
$$h_5(x):=f(x)+3f\left(\frac{1-x}{3}\right),\quad h_6(x):=2f\left(\frac{x}{2}\right)+f\left(\frac{11}{16}-x\right)+f\left(\frac{5}{16}\right).$$

Then, we have

(a) By

$$h_1'(x)=\frac{-5(2x-1)(112x^2-112x+1)}{2a(x)a(1-x)\{a^2(x)+a^2(1-x)\}}=\begin{cases} >0, & \frac{11}{16}\le x<\frac{1}{2}+\frac{3\sqrt{21}}{28} \\ <0, & \frac{1}{2}+\frac{3\sqrt{21}}{28}<x<1 \end{cases} \tag{2.20}$$

Thus,

$$\min_{\frac{11}{16}\le x<1}h_1(x)=\min\left\{h_1\left(\frac{11}{16}\right),h_1(1)\right\}=h_1\left(\frac{11}{16}\right)=\frac{\sqrt{11}+\sqrt{3}}{6}. \tag{2.21}$$

(b) By

$$h_2'(x)=\frac{-5(32x-11)(9728x^2-6688x+729)}{2048a(x)a\left(\frac{11}{16}-x\right)\left\{a^2(x)+a^2\left(\frac{11}{16}-x\right)\right\}}=\begin{cases} >0, & \frac{6}{16}<x<\frac{11}{32}+\frac{29\sqrt{19}}{608} \\ <0, & \frac{11}{32}+\frac{29\sqrt{19}}{608}<x<\frac{11}{16} \end{cases} \tag{2.22}$$

Thus,

$$\min_{\frac{6}{16}<x\le\frac{11}{16}}h_2(x)=\min\left\{h_2\left(\frac{6}{16}\right),h_2\left(\frac{11}{16}\right)\right\}=h_2\left(\frac{11}{16}\right)=\frac{\sqrt{11}+\sqrt{3}}{6}. \tag{2.23}$$

(c) By

$$h_3'(x)=\frac{-5(3x-2)(40x^3+36x^2-50x+1)}{2a(1-x)a\left(\frac{x}{2}\right)\left\{a^2(1-x)+a^2\left(\frac{x}{2}\right)\right\}}=\begin{cases}>0, & \frac{11}{16}<x<t_1\approx 0.741120\\ <0, & t_1<x<1\end{cases} \tag{2.24}$$

Thus,

$$\min_{\frac{11}{16}\le x<1} h_3(x)=\min\left\{h_3\left(\frac{11}{16}\right),h_3(1)\right\}=2f\left(\frac{1}{2}\right)=\frac{\sqrt{6}}{3}. \tag{2.25}$$

(d) By

$$h_4'(x)=\frac{-5(16x-3)(1536x^2-576x+343)}{128a(x)a\left(\frac{6}{16}-x\right)\left\{a^2(x)+a^2\left(\frac{6}{16}-x\right)\right\}}=\begin{cases}>0, & 0\le x<\frac{3}{16}\\ <0, & \frac{3}{16}<x<\frac{6}{16}\end{cases} \tag{2.26}$$

Thus,

$$\min_{\frac{5}{16}\le x\le\frac{6}{16}} h_4(x)=h_4\left(\frac{6}{16}\right)=f\left(\frac{6}{16}\right)+2f\left(\frac{5}{16}\right)=\frac{\sqrt{21}}{14}+\frac{\sqrt{3}}{3}. \tag{2.27}$$

$$\max_{0<x<\frac{5}{16}} h_4(x)=h_4\left(\frac{3}{16}\right)=2f\left(\frac{3}{16}\right)+2f\left(\frac{5}{16}\right)=\frac{\sqrt{51}}{17}+\frac{\sqrt{3}}{3}. \tag{2.28}$$

(e) By

$$h_5'(x)=\frac{5(4x-1)(1280x^3-4656x^2+4680x-1331)}{162a(x)a\left(\frac{1-x}{3}\right)\left\{a^2(x)+a^2\left(\frac{1-x}{3}\right)\right\}}=\begin{cases}>0, & 0<x<\frac{1}{4}\\ <0, & \frac{1}{4}<x<t_2\approx 0.49498868\\ >0, & t_2<x<t_3\approx 0.96454164\\ <0, & t_3<x<1\end{cases} \tag{2.29}$$

Thus,

$$\min_{0<x\le\frac{5}{16}} h_5(x)=\min\left\{h_5(0),h_5\left(\frac{5}{16}\right)\right\}=h_5(0)=3f\left(\frac{1}{3}\right)=\frac{3}{\sqrt{11}}. \tag{2.30}$$

$$\max_{\frac{5}{16}\le x<1} h_5(x)==\max\left\{h_5\left(\frac{5}{16}\right),h_5(t_3)\right\}=h_5(t_3)=f(t_3)+3f\left(\frac{1-t_3}{3}\right)\approx 1.065646364. \tag{2.31}$$

where t_3 is the root of the following cubic equation

$$1280x^3-4656x^2+4680x-1331=0$$

closest to the right endpoint 1 in the interval [0,1].

(f) By

$$h_6'(x)=\frac{-5(24x-11)(2560x^3+5184x^2-1400x+729)}{2048a\left(\frac{x}{2}\right)a\left(\frac{11}{16}-x\right)\left\{a^2\left(\frac{x}{2}\right)+a^2\left(\frac{11}{16}-x\right)\right\}}=\begin{cases}>0, & 0<x<\frac{11}{24}\\ <0, & \frac{11}{24}<x<1\end{cases} \tag{2.32}$$

Thus,

$$\min_{\frac{10}{16}\le x\le\frac{11}{16}} h_6(x)=h_6\left(\frac{11}{16}\right)=2f\left(\frac{11}{32}\right)+f\left(\frac{5}{16}\right)=\frac{\sqrt{319}}{29}+\frac{\sqrt{3}}{6}. \tag{2.33}$$

$$\max_{0<x\le\frac{6}{16}} h_6(x) = h_6\left(\frac{6}{16}\right) = 2f\left(\frac{3}{16}\right) + 2f\left(\frac{5}{16}\right) = \frac{\sqrt{51}}{17} + \frac{\sqrt{3}}{3}. \qquad (2.34)$$

Comparing the minimal values and maximal values in (2.21) - (2.34), and combine (2.4) with (2.19), we get (1.4).

The proof is finished.

III. REFERENCES

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