International Journal of Research in Engineering and Science (IJRES)
ISSN (Online): 2320-9364, ISSN (Print): 2320-9356
www.ijres.org Volume 9 Issue 1 ǁ 2021 ǁ PP. 17-21

Existence Result for Fourth Order Random Differential Equation

D. S. Palimkar
Department of Mathematics, Vasantrao Naik College, Nanded
PIN-431603 (M.S.) INDIA

ABSTRACT
In this paper, we discuss the boundary value problem for fourth order random differential equation. Prove existence of random solution by applying random fixed point theorem under generalized Lipschitz and Caratheodory conditions.
KEYWORDS: *Random differential equation,Green's function, fixed point theorem, caratheodory condition.*
2000MathematicsSubjectClassifications: 34B14, 34B15,47H10.

Date of Submission: 12-01-2021 Date of acceptance: 27-01-2021

I. DESCRIPTION OF THE PROBLEM

We consider boundary value problem of fourth order random differential equation

$$\begin{aligned} &x''''(t,\omega)+f(x,t,\omega)=0\ ,\ 0<t<1 \\ &x''(0,\omega)=0\ ,\ x'(1,\omega)=x''(\eta,\omega)=x''(\eta,\omega)=\alpha \end{aligned} \tag{1.1}$$

for all $\omega\in\Omega$, where $f: J\times R\times\Omega\to R$.

By a random solution of the problem(1.1), we mean a measurable function $x:\Omega\to AC^1(J,R))$ that satisfies the equations in (1.1), where $AC^1(J,R)$ is the space of continuous real-valued functions.

The classical boundary value problem (1.1) is studied by many authors such as in Lakshmikantham and Leela[5], ,Nieto [6], Yao [8] but in random is rare as in Bharucha-Reid [1,2] and Itoh [5].Here, we discuss the random boundary value problem(1.1) for existence of random solution under under generalized Lipschitz and Caratheodory conditions applying the random fixed point theorem.

II. AUXILIARY RESULTS

We use the following random fixed point theorem to prove main result.

Theorem2.1.(Dhage[3,4]): Let U be a non-empty, open and bounded subset of the separable Banach space E such that $0\in U$ and let $Q:\Omega\times\bar{U}\to E$ be a compact and continuous random operator. Further suppose that there does not exists an $u\in\partial U$ such that $Q(\omega)u=ux$ for al l $\omega\in\Omega$, where α >1 and ∂U is the boundary of U in E . Then the random equation $Q(\omega)x=x$ has a random solution, i.e., there is a measurable function $\xi:\Omega\to E$ such that $Q(\omega)\xi(\omega)=\xi(\omega)$ for all $\omega\in\Omega$.

In application form

Corollary2.1. Let $B_r(0)$ and $\overline{B_r}(0)$ be the open and closed balls centered at origin of radius r in the separable Banach space E and let $Q:\Omega\times\overline{B_r}(0)\to E$ be a compact and continuous random operator. Further suppose that there does not exists an $u\in E$ with $\|u\|=r$ such that $Q(\omega)u=\alpha u$ for all $\omega\in\Omega$, where $\alpha>1$. Then the random equation $Q(\omega)x=x$ has a random solution, i.e., there is a measurable function $\xi:\Omega\to\overline{B_r}(0)$ such that Q(ω)ξ (ω)=ξ (ω)forall $\omega\in\Omega$.

Also, we need the following theorem and lemma.

Theorem2.2.(Caratheodory): Let $Q:\Omega\times E\to E$ be a mapping such that $Q(x,.)$ is measurable for all $x\in E$ and $Q(\omega,.)$ is continuous for all $\omega\in\Omega$. Then the map $(\omega,x)\to Q(\omega,x)$ is jointly measurable.

The following lemma is used to prove main result.

Lemma2.1.[6]. For any real number $m>0$ and $\sigma \in L^1(J,R)$, x is a solution to the differential equation

$$x''(t)+m^2x(t)=\sigma(t) \quad a.e.\ t\in Jm$$
$$x(0)=x(2\pi),\ x'(0)=x'(2\pi)$$

(2.1)

if and only if it is a solution of the integral equation

$$x(t)=\int_0^{2\pi} G_m(t,s)\sigma(s)ds \tag{2.2}$$

Where,

$$G_m(t,s) = \frac{1}{2m(e^{2m\pi}-1)}\left[e^{m(t-s)}+e^{m(2\pi-t+s)}\right], \quad if \quad 0\le s\le t\le 2\pi,$$
$$= \frac{1}{2m(e^{2m\pi}-1)}\left[e^{m(s-t)}+e^{m(2\pi-s+t)}\right], \quad if \quad 0\le t< s\le 2\pi$$

(2.3)

The Green's function G_m is continuous and nonnegative on $J\times J$ and the numbers

$$\alpha = \min\left\{\left|G_m(t,s)\right| : t,s\in[0,2\pi]\right\} = \frac{e^{m\pi}}{m(e^{2m\pi}-1)}$$

and

$$\beta = \max\left\{\left|G_m(t,s)\right| : t,s\in[0,2\pi]\right\} = \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}$$

exist for all positive real number m .

Definition2.1.A function $f: J\times R\times\Omega\to R$ is called random carathedory if

i) the map $(t,\omega)\to f(t,x,\omega)$ is jointly measurable for all $x\in R$, and

(ii)the map $x\to f(t,x,\omega)$ is continuous for all $t\in J$ and $\omega\in\Omega$.

Definition2.2.A function $f: J\times R\times\Omega\to R$ is called L^1- random caratheodory if (iii) for each real number r>0 there is a measurable and bounded function $q_r:\Omega\to L^1(J,R)$ such that

$$\left|f(t,x,\omega)\right|\le q_r(t,\omega) \quad a.e.\ t\in J \text{ ,for all } \omega\in\Omega \text{ and } x\in R \text{ with } |x|\le r.$$

III. MAIN EXISTENCE RESULT

We have the following assumptions

(A_1). The function f_m is random caratheodory on $J\times R\times\Omega$.

(A_2). There exists a measurable and bounded function $\gamma:\Omega\to L^2(J,R)$ and a continuous and non-decreasing function $\psi: R_+\to(0,\infty)$ such that

$$\left|f_m(t,x,\omega)\right|\le\gamma(t,\omega)\psi(|x|) \quad a.e.t\in J$$

for all $\omega\in\Omega$ and $x\in R$.

Theorem3.1. Assume that (A_1)-(A_2) holds. Suppose that there exists a real number $r>0$ such that

$$r>\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\left\|\gamma(\omega)\right\|_{L^1}\psi(r) \tag{3.1}$$

for all $\omega\in\Omega$ .Then the random problem(1.1) has a random solution defined on J .

Proof: Set $E = C(J, R)$ and define a mapping $Q : \Omega \times E \to E$ by

$$Q(\omega)x(t) = \int_0^{2\pi} G_m(t,s) f_m(s, x(s,\omega), \omega) ds$$

for all $t \in J$ and $\omega \in \Omega$ .The map $t \to G_m(t,s)$ is continuous on J , $Q(\omega)$ defines a mapping $Q : \Omega \times E \to E$. Define a closed ball $\bar{B}_r(0)$ in E centered at origin 0 of radius r , where the real number r satisfies the inequality (3.3). we show that Q satisfies all the conditions of Corollary2.1on $\bar{B}_r(0)$.

First, we show that Q is a random operator on $\bar{B}_r(0)$. Since $f_m(t,x,\omega)$ is random Carath'eodory, the map $\omega \to f_m(t,x,\omega)$ is measurable in view of Theorem 2.2. Similarly, the product $G_m(t,s) f_m(s, x(s,\omega), \omega)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$\omega \to \int_0^{2\pi} G_m(t,s) f_m(s, x(s,\omega), \omega) ds = Q(\omega)x(t)$$

is measurable. As a result, Q is a random operator on $\Omega \times \bar{B}_r(0)$ into E .

Then, show that the random operator $Q(\omega)$ is continuous on $\bar{B}_r(0)$. Let $\{x_n\}$ be a sequence of points in $\bar{B}_r(0)$ converging to the point x in $\bar{B}_r(0)$ .Then ,it is enough to prove that $\lim_{n\to\infty} Q(\omega)x_n(t) = Q(\omega)x(t)$ for all $t \in J$ and $\omega \in \Omega$. By dominated convergence theorem, we obtain,

$$\lim_{n\to\infty} Q(\omega)x_n(t) = \lim_{n\to\infty} \int_0^{2\pi} G_m(t,s) f_m(s, x_n(s,\omega), \omega) ds$$

$$= \lim_{n\to\infty} \int_0^{2\pi} G_m(t,s) \lim_{n\to\infty} [f_m(s, x_n(s,\omega), \omega)] ds$$

$$= \int_0^{2\pi} G_m(t,s) [f_m(s, x(s,\omega), \omega)] ds$$

$$= Q(\omega)x(t)$$

for all $t \in J$ and $\omega \in \Omega$. This shows that $Q(\omega)$ is a continuous random operator on $\bar{B}_r(0)$.

Now,we show that $Q(\omega)$ is a compact random operator on $\bar{B}_r(0)$. To complete the proof, if it is enough to prove that $Q(\omega)(\bar{B}_r(0))$ is uniformly bounded and equi-continuous set in E for each $\omega \in \Omega$. Since the map $\omega \to \gamma(t,\omega)$ is bounded and $L^2(J,R) \subset L^1(J,R)$,by hypothesis (A_2),there is constant c such that $\|\gamma(\omega)\|_{L^1} \le c$ for all $\omega \in \Omega$ .Let $\omega \in \Omega$ be fixed. Then for any $x : \Omega \to \bar{B}_r(0)$,one has

$$|Q(\omega)x(t)| \le \int_0^{2\pi} G_m(t,s) |f_m(s, x(s,\omega), \omega)| ds$$

$$\le \int_0^{2\pi} G_m(t,s) \gamma(s,\omega) \psi(|x(s,\omega)|) ds$$

$$\le \frac{e^{2m\pi} + 1}{2m(e^{2m\pi} - 1)} \left(\int_0 \gamma(s,\omega) ds \right) \psi(r)$$

$$\leq K_1$$

for all $t \in J$, where $K_1 = \dfrac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} c\psi(r)$ .This shows that $Q(\omega)(\bar{B}_r(0))$ is a uniformly bounded subset of E for each $\omega \in \Omega$.

Then, we show that $Q(\omega)(\bar{B}_r(0))$ is an equi-continuous set in E .Le $x \in \bar{B}_r(0)$ be arbitrary. Then for any $t_1, t_2 \in J$, one has $|Q(\omega)x(t_1) - Q(\omega)x(t_2)| \leq$
$\int_0^{2\pi} |G_m(t_1,s) - G_m(t_2,s)| |f_m(s, x(s,\omega), \omega)| ds$

$$\leq \int_0^{2\pi} (|G_m(t_1,s) - G_m(t_2,s)|^2 ds)^{\frac{1}{2}} (\int_0^{2\pi} |\gamma(s,\omega)|^2 ds)^{\frac{1}{2}} \psi(r).$$

Hence for all $t_1, t_2 \in J$, $|Q(\omega)x(t_1) - Q(\omega)x(t_2)| \to 0$ as $t_1 \to t_2$, uniformly for all $x \in \bar{B}_r(0)$ .Therefore,
$Q(\omega)(\bar{B}_r(0))$ is an equi-continuous set in E. As $Q(\omega)(\bar{B}_r(0))$ is uniformly bounded and equi-continuous, it is compact by Arzel -Ascoli theorem for each $\omega \in \Omega$. Consequently, $Q(\omega)$ is a completely continuous random operator on $\bar{B}_r(0)$.

Lastly, prove that there does not exist an $u \in E$ with $\|u\| = r$ such that $Q(\omega)u(t) = \alpha\, u(t,\omega)$ for all $t \in J$ and $\omega \in \Omega$, where $\alpha >$ 1. Suppose not. Then there exists an element $u \in E$ satisfying $Q(\omega)u(t) = \alpha\, u(t,\omega)$ for some $\omega \in \Omega$ .Let $\lambda = \dfrac{1}{\alpha}$ .Then $\lambda < 1$ and $\lambda Q(\omega)u(t) = u(t,\omega)$ for some $\omega \in \Omega$.
we have

$$\begin{aligned}
|\mathrm{u}(\mathrm{t},\omega)| &\leq \lambda |Q(\omega)\mathrm{u}(\mathrm{t})| \\
&\leq \int_0^{2\pi} G_m(t,s) |f_m(s, u(s,\omega), \omega)| ds \\
&\leq \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \int_0^{2\pi} \gamma(s,\omega)\psi(\|u(\omega)\|) ds \\
&\leq \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \|\gamma(\omega)\|_{L^1} \psi(\|u(\omega)\|)
\end{aligned}$$

for all $t \in J$.

Taking supremum ,we get,

$$\|u(\omega)\| \leq \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \|\gamma(\omega)\|_{L^1} \psi(\|u(\omega)\|)$$

or

$$r \leq \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \|\gamma(\omega)\|_{L^1} \psi(r)$$

for some $\omega \in \Omega$. This contradicts to the condition (3.3).

Thus, all the conditions of Corollary 2.1 are satisfied. Hence equation $Q(\omega)x(t) = x(t,\omega)$ has a random solution in $\bar{B}_r(0)$,such that $Q(\omega)\xi(t) = \xi(t,\omega)$ for all $t \in J$ and $\omega \in \Omega$ where $\xi : \Omega \to \bar{B}_r(0)$. Hence, the random problem (1.1) has a random solution defined on J .

ACKNOWLEDGEMENT

The paper is outcome result of Minor Research Project funded by Swami Ramanand Teerth Marathwada University,Nanded[MS] India.

REFERENCES

[1]. Bharucha-Reid A.T., Random Integral Equations ,Academic Press , New York,1972.

[2]. Bharucha-Reid A. T., Fixed Point Theorems in Probabilistic Analysis, Bull. Amer. Math. Soc., (1976) 82, 641-657.

[3]. .B. C. Dhage , J. Henderson, Existence theory for Nonlinear Functional Boundary value Problems ,Electronic Journal of Qualitative Theory of Differential Equations,1(2004),1-15.

[4]. B. C. Dhage, A random version of a Schaefer type fixed point theorem with applications to functional random integral equations, Nonlinear Funct. Anal.Appl 9 (2004),389-403.

[5]. S. Itoh, Random fixed point theorems with applications to random differential equations Banach spaces, J. Math. Anal. Appl. 67 (1979),261-273.

[6]. V. Lakshmikantham ,S. Leela, Remarks on first and second order periodic boundary value problem, Nonlinear Anal. 8 (1984), 281-287.

[7]. J. J. Nieto, Nonlinear second order periodic value problems with Carath'eodory functions, Appl.Anal.34(1989),111–128.

[8]. D. S. Palimkar, Existence theory of random differential equation, Inter. Journ.of Sci. and Res. Pub.,Vol.2, 7,2012,1-6.

[9]. D.S. Palimkar ,Existence Theory of second order random differential equation,Global Journal of Mathematical sciences Theory and Practical ,Volume -4 ,No.4 (2012).

[10]. Q. Yao, Positive solutions of nonlinear second-order periodic boundary value problems,Appl. Math. Lett. 20 (2007), 583-590.