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Multipartite entanglement in interacting "atom + quantized field" system

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Abstract
The interaction of a three-level atom with two modes of quantized field is investigated analytically. Strong entanglement between different parts of the studied multi-partite system is found and described both qualitatively and quantitatively. A possibility to control and tailor the observed entanglement either by features of the acting fields or by initial parameters of the scheme is demonstrated. The considered system can be used as a controlled transmission unit with managed multi-partite entanglement resource.

Keywords:*non-classical light, interaction of atoms with quantum field, entanglement, Schmidt parameter, squeezed fields, mixed states.*

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I. INTRODUCTION

An important part of modern information technology appears to be the development of algorithms providing transfer, storage and extraction of quantum information. Such protocols are usually based on quantum states of light that recently become available in experiment. Now different quantum states of light can be generated: Fockstates, coherent states with small number of photon, squeezed vacuum light, Schroedinger cat states etc [1-8].Among them squeezed states of light appears to be the most attractive due to their unique properties that are very perspective for many important and interesting practical applications. Such non-classical states of light can be produced in a parametric down conversion process and are known to be characterized by strong correlations between photons [9-11]. In addition such light is characterized by suppressed variance of one of the field quadratures which makes possible the high resolution measurements with noise reduction beyond the standard quantum limit. Now it is possible to generate squeezed states of light with huge mean number of photons up to 10^{15} per mode[12]. In this sense it is possible to refer such states of light to as macroscopic quantum states.

From the point of view of purposes for information technology, the transfer of quantum information seems to be better to provide using such nonclassical states of light while the storage is more convenient to beorganized on the basis of the so called atomic qubits. Different systems can be considered as qubits including superposition of states in atoms, semiconductors and especially semiconductor quantum wells, superconducting Josephson qubitsetc [13,14]. To perform good transfer and exchange of quantum information including the phase, perfect interface between the quantum light and atomic qubits is strongly required. The key point of such protocols consists in strong entanglement between atomic and field subsystems which takes place during their interaction. A simple example of the entanglement is the well-known Jaynes-Cummings model [15] describing the interaction of two-level atom with one mode of quantized field. The advantage of such quantum entanglement consists in the possibility to extract information of one part of the system by measuring another one. And the degree of entanglement can be simply calculated for example using the so called Schmidt parameter [16, 17]. However more complicated entangled systems are usually of great interest since multiport quantum control can be possible in such case. For example a three-level atom interacting with two modes of quantized field can be considered as important controlled transmission link. Moreover atomic systems involved in different physical processes can be often well described using the 3-level scheme [15]. However for such multipartite systems it is very difficult to establish the entanglement since there are no methods to characterize correctly the entanglement even for 3 interacting subsystems.

In this paper an analytical solution for a 3-level atom interacting with two modes of quantized field is presented. The arising multipartite entanglement is analyzed both qualitatively and quantitatively. The possibility of strong entanglement between atomic and field subsystem is demonstrated. Moreover the entanglement between initially independent field modes is found to be induced due to their interaction via the atomic subsystem. Methods to increase and control the observed entanglement are suggested. The entanglement resource found in such multipartite systems is shown to be very promising for many perspective applications in information technologies.

II. THEORETICAL APPROACH

We solve analytically the problem of resonant interaction of three-level atomic system with two modes of quantum electromagnetic field. The atom is considered in so-called Λ-configuration with possible transitions induced resonantly by the first field (with frequency ω_1) between the lowest and upper atomic levels and by the second field (with frequency ω_2) - between the second low level and the upper one.

The interaction of the atom with two modes of quantized electromagnetic field is studied in the frame of the non-stationary Schrödinger equation with both quantum field degrees of freedom being taken into account:

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(\widehat{H}_{at} + \widehat{H}_{field\ 1} + \widehat{H}_{field\ 2} + \widehat{W}_1 + \widehat{W}_2\right)\Psi \tag{1}$$

The total Hamiltonian consists of the atomic and field interaction-free Hamiltonians $\widehat{H}_{at}$ *and* $\widehat{H}_{field\ 1,2}$ respectively and the interaction terms $\widehat{W}_i = d\varepsilon_{0i}q_i$ taken in the dipole approximation. Here $\boldsymbol{d} = e\boldsymbol{r}$ stands for the operator of atomic dipole moment, q_i - for the dimensionless electromagnetic field in each mode and ε_{0i} is a correspondent normalization constant which characterises the coupling strength between the atom and a field and depends on the interaction volume L^3: $\varepsilon_{0i} = \sqrt{\frac{4\pi\hbar\omega_i}{L^3}}$. It should be noticed that the efficiency of atom-field interaction can be significantly increased by using small micricavity which allows to observe experimentally even vacuum Rabi oscillations [3]. For our model we suppose the strength of atomic-field interaction which is determined by the mean photon density $< N >/ L^3$ large enough to provide the characteristic Rabi oscillations significantly faster than any decoherence processes in the system.

The Eq. (1) is solved by expansion of the time-dependent wave function of the total system over the interaction-free atomic eigenfunctions $\varphi_n(\vec{r})$ and Fock states of each field mode $\Phi_m(q_i)$:

$$\psi(\vec{r}, q_1, q_2, t) = \sum_{nkm} C_{nkm}(t)\,\varphi_n(\vec{r})\Phi_k(q_1)\Phi_m(q_2) \times exp\left(-\frac{i}{\hbar}E_{nkm}\,t\right) \tag{2}$$

where the energyE_{nkm} is given by $E_{nkm} = E_n + \hbar\omega_1(k + \frac{1}{2}) + \hbar\omega_2(m + \frac{1}{2})$.

Substitution of solution (2) into Eq. (1) with using the rotative-wave approximation leads to the system of differential equations for the probability amplitudes $C_{nkm}(t)$ to find the atom in eigenstate φ_n and k and m photons in one and another quantum fields respectively:

$$\begin{cases} i\hbar\dot{C}_{1(k+1)m} = -d_{13}\varepsilon_{01}C_{3km}\sqrt{\dfrac{k+1}{2}} \\ i\hbar\dot{C}_{2k(m+1)} = -d_{23}\varepsilon_{02}C_{3km}\sqrt{\dfrac{m+1}{2}} \\ i\hbar\dot{C}_{3km} = -d_{13}\varepsilon_{01}C_{1(k+1)m}\sqrt{\dfrac{k+1}{2}} - d_{23}\varepsilon_{02}C_{2k(m+1)}\sqrt{\dfrac{m+1}{2}} \end{cases} \tag{3}$$

Here $d_{ij} =< \varphi_i|er|\varphi_j >$ is the matrix element of the electron dipole moment for transition between atomic levels with numbers *i* and *j*. The system (3) is solved analytically for each coupled triad of probability amplitudes:

$$
\begin{cases}
C_{1(k+1)m}(t) = \gamma_{km} + \dfrac{d_{13}\varepsilon_{01}}{\xi_{km}}\sqrt{\dfrac{k+1}{2}}\left(\alpha_{km}\, exp\left(\dfrac{i}{\hbar}\xi_{km}t\right) - \beta_{km}\, exp\left(-\dfrac{i}{\hbar}\xi_{km}t\right)\right) \\
C_{2k(m+1)} = -\dfrac{d_{13}\varepsilon_{01}}{d_{23}\varepsilon_{02}}\gamma_{km} + \dfrac{d_{23}\varepsilon_{02}}{\xi_{km}}\sqrt{\dfrac{m+1}{2}}\left(\alpha_{km}\, exp\left(\dfrac{i}{\hbar}\xi_{km}t\right) - \beta_{km}\, exp\left(-\dfrac{i}{\hbar}\xi_{km}t\right)\right) \\
C_{3km} = \alpha_{km}\, exp\left(\dfrac{i}{\hbar}\xi_{km}t\right) + \beta_{km}\, exp\left(-\dfrac{i}{\hbar}\xi_{km}t\right)
\end{cases} \tag{4}
$$

with notation $\xi_{km}^2 = \frac{1}{2}\left(d_{13}^2\varepsilon_{01}^2(k+1) + d_{23}^2\varepsilon_{02}^2(m+1)\right)$.

The coefficients $\alpha_{km}, \beta_{km}, \gamma_{km}$ can be found from the initial conditions for the atom and fields

$$\psi(\vec{r}, q_1, q_2, t=0) = \sum_{nkm} C_{nkm}^0\, \varphi_n(\vec{r})\Phi_k(q_1)\Phi_m \tag{5}$$

and are given by:

$$
\begin{cases}
\alpha_{km} = \dfrac{1}{2}C_{3km}^0 + \dfrac{1}{2\sqrt{2}\xi_{km}} \times (d_{13}\varepsilon_{01}\sqrt{k+1}C_{1(k+1)m}^0 + d_{23}\varepsilon_{02}\sqrt{m+1}C_{2k(m+1)}^0) \\
\beta_{km} = \dfrac{1}{2}C_{3km}^0 - \dfrac{1}{2\sqrt{2}\xi_{km}} \times (d_{13}\varepsilon_{01}\sqrt{k+1}C_{1(k+1)m}^0 + d_{23}\varepsilon_{02}\sqrt{m+1}C_{2k(m+1)}^0) \\
\gamma_{km} = \dfrac{d_{23}\varepsilon_{02}\sqrt{m+1}}{2\xi_{km}^2} \times (-d_{13}\varepsilon_{01}\sqrt{k+1}C_{1(k+1)m}^0 + d_{23}\varepsilon_{02}\sqrt{m+1}C_{2k(m+1)}^0)
\end{cases} \tag{6}
$$

As an initial condition we suppose the atom to be generally in a some superposition of its eigenstatesand consider the quantum fields to be in coherent state with small number of photons or in a squeezed vacuum state. The coherent state $|\alpha >$ with phase ϑ can be expanded over different Fock state by following way:

$$|\alpha_\vartheta >= \sum_n exp\left(-\frac{|\alpha|^2}{2}\right)\frac{|\alpha|^n\, exp(in\theta)}{\sqrt{n!}}\Phi_n \tag{7}$$

This expression leads to the Poisson distribution over Fock states with mean number of photons equal to $< N >= |\alpha|^2$ and photon number variance $D_n =< N >$. As for the squeezed vacuum state, it is characterized by much more broader distribution over Fock states with only even numbers contributing to field wave function [18]:

$$\Psi_{sq} = \sum_n C_{2n}\,\Phi_{2n} \tag{8}$$

$$C_{2n} = (-1)^n\sqrt{\frac{2\gamma}{1+\gamma^2}}\frac{\sqrt{(2n)!}}{2^n n!}\left(\frac{1-\gamma^2}{1+\gamma^2}\right)^n \tag{9}$$

Here γ is the squeezing parameter which determines the mean photon number as $< N >= \frac{1}{4}\left(\gamma - \frac{1}{\gamma}\right)^2$. And the photon number variance appears to be extremely large in this case: $D_n = 2 < N >^2 + 2 < N >$.

Using the found time-dependent wave function of the total system (2) it is possible to get any information about the atomic and quantum field subsystems and their interaction including the entanglement. For example, the time-dependent population of atomic levels $P_n(t)$ and of different Fock states for each field mode$W_k(t)$ and $\widetilde{W}_m(t)$ can be found respectively as following:

$$P_n(t) = \sum_k\sum_m |C_{nkm}|^2 \tag{10}$$

$$W_k(t) = \sum_n\sum_m |C_{nkm}|^2 \tag{11}$$

$$\widetilde{W}_m(t) = \sum_n\sum_k |C_{nkm}|^2 \tag{12}$$

The main goal of this paper is to examine in details the entanglement arising in the studied multipartite system.

III. RESULTS AND DISCUSSION

There are many ways to characterize the degree of entanglement between two subsystems [16, 17, 19, 20] while the entanglement between all components in a three-partite system is rather difficult to describe quantitatively.Two analyze the entanglement in multipartite system two main different strategies can be used. The first one means to characterize one of interacting subsystems independently on the behavior and dynamics of the others. Actually, such approach supposes to average over all variables except the considered subsystem and to find its entanglement with the other part of the whole system. Another way consists in the so called conditional measurement which describes the subsystem of interest under the condition that the other part of the whole system is found in a certain state. These two different ways make possible to characterize the entanglement from different points and are both discussed below.

3.1 Evolution of the atom and entanglement between atomic and field subsystems

If we examine the evolution of atomic system independently on the states of both quantum fields we should average over the field variables.Then we see that in a most general case the atom is found in a mixed state rather than in a pure state due to its interaction with quantized fields and is characterized by some density matrix with matrix elements given by:

$$\rho_{ij}^{at} = \sum_{n,k} C_{ink}\, C_{jnk}^{*}\, exp\left(-\frac{i\left(E_j - E_i\right)t}{\hbar}\right) \tag{13}$$

Diagonal elements of this matrix coincide exactly with time-dependent populations of atomic levels (10) while the off-diagonal elements keep the information about phase coherence in the formed atomic state.To characterize the entanglement between atomic and field subsystems we can use the Schmidt parameter which is given by [16, 17, 21]:

$$K = [Sp(\rho_{at}^2)]^{-1} \tag{14}$$

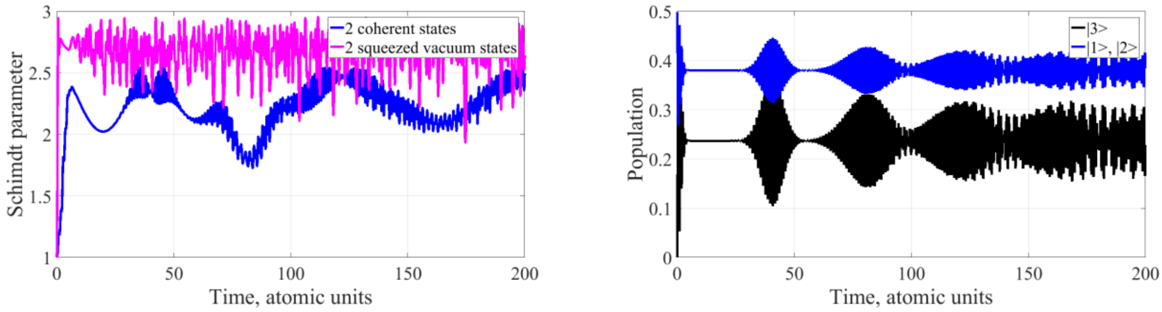
Using the matrix elements (13) for our system this parameter can be calculated as following:

$$K = [\sum_{ij} |\rho_{ij}^{at}|^2]^{-1} \tag{15}$$

It is seen that both the diagonal and off-diagonal matrix elements contribute to the value of the Schmidt parameter. The more is this parameter the higher is the degree of entanglement. The minimal value of this parameter is equal to 1 and corresponds to the case of fully independence between the atom and fields. Under such conditions simple factorization of atomic and field states take place and the atom is found in a pure state. The maximal value of the Schmidt parameter depends on the number of the basis atomic eigenstates and is equal to 3 for our system. It can be under stood that the maximal entanglement corresponds to the case when the atom is in a mixed state with all off-diagonal matrix elements equal to zero. Such state can be reffered to as the most possible mixed state. So, the Schmidt parameter can be used also to characterize how mixed is the state. At the same time its inverse value can be considered as the degree of purity of atomic state. Indeed, if the Schmidt parameter is equal to 1, the atomis density matrix describes a pure state since $Sp(\rho_{at}^2) = Sp\left(\rho_{at}\right) = 1$. It should be noticed that in a pure state the off-diagonal matrix elements are always non-zero.

For our multipartite system the Schmidt parameter characterizing the entanglement between the atom and two fields is presented at Fig. 1 for two different cases: when initially two coherent or two squeezed vacuum states interact with the atom. The atom initially is supposed tooccupy the coherent superposition of two low atomic levels with some relative phase between them:

$$\psi(\vec{r}, t = 0) = \frac{\varphi_1 + e^{i\Delta}\varphi_2}{\sqrt{2}} \tag{16}$$



a) b)

Fig. 1.Dynamics of the Schmidt parameter K obtained for two coherent states and two squeezed vacuum states (a) and regimes of "collapse" and "revival" of atomic transitions (b) presented in the case of initial atomic state $\frac{\varphi_1+e^{i\Delta}\varphi_2}{\sqrt{2}}$ for $\Delta=\frac{\pi}{2}$ and mean photon number $\langle N\rangle = 10$.

In this case it is found that the highest degree of entanglement is reached for $\Delta=\frac{\pi}{2}$since all of quaienergy states of the system are initially populated at such initial condition [22]. The time-dependent behaviour of the Schmidt parameter is shown to be very different for two initially coherent or squeezed vacuum fields. For two coherent fields the Schmidt parameter reveals the change between smooth and oscillating behaviour which results from the arising effects of "collapse" and "revival" of atomic transitions found in [22] and demonstrated at Fig. 1b for the populations of atomic states.

In the case when two squeezed vacuum fields interact with atomic system, such regimes do not take place and the evolution of the Schmidt parameter is characterized by more or less chaotic iscillations arising due to a lot of Fock states with different numbers that contribute to such initial squeezed state. Moreover the mean value of the Schmidt parameter averaged over time is found to be significantly larger than in the case of coherent fields and is rather close to its maximal value equal to 3 in this case. Thus there is the possibility to achieve and shoose different regimes of entanglement between atomic and field subsystems.

3.2 Entanglement between the atom and one of the two interacting fields

For some practical purposes and applications it is important to manage the entanglement between an atom and one mode of electromagnetic field while the second mode is used as a control field. To calculate the entanglement in this case the third subsystem (the control field) should be excluded either by averaging over all its possible states or by fixing a certain state for this system. Let us consider the second way. In this case it is very convenient to fix a certain number of photons in the control field and calculate the entanglement under such condition. To do this we project the total wave function of three-partite system onto the Fock state with m photons of the second field $\widetilde{\Phi}_m(q_2)$ and then obtain the wave function describing a bi-partite system namely"atom+ one mode of quantized field":

$$\psi(\vec{r},q_1)=\langle\widetilde{\Phi}_m(q_2)|\psi_{total}\rangle \tag{17}$$

Further procedure of calculation of entanglement is simple and consists in obtaining the Schmidt parameter by usual way. Using the projected wave function (17), it is possible to construct the density matrix of thebi-partite system

$$\rho=|\psi\rangle\langle\psi| \tag{18}$$

andthereduced atomic density matrix which matrix elements are given by:

$$(\rho_{red})_{n_1}^{n_2}=\sum_k C_{n_1km}C_{n_2km}^{*}\,exp\left(-\frac{i\left(E_{n_1}-E_{n_2}\right)t}{\hbar}\right) \tag{19}$$

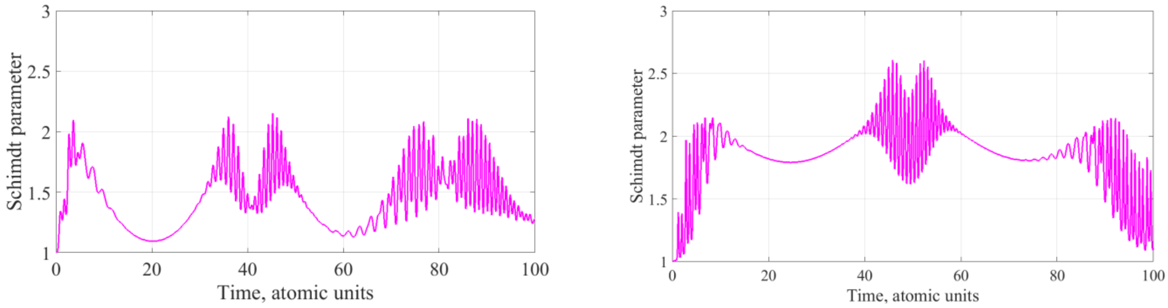
Here the indices$n_1$and $n_2$take the values from 1 to 3 corresponding to the numbers of atomic levels.

The final step is to characterize the entanglement quantitatively by calculating the Schmidt parameter:

$$K=\frac{1}{Sp((\rho_{red})^2)} \tag{20}$$

Note that in contrast to Eq. (13) the summation in (19) is performed over k only while the number of photons m in the second field is supposed to be fixed. It means that we calculate the entanglement under the condition that m photons are found in the second quantized field. Such approach has physical reasons since in experiment the number of photons in a field can be measured directly or the procedure of post-selection can be used.

Under such conditions the Schmidt parameter calculated for the atom and one mode of quantized field using Eq. (20) is presented as a function of time at Fig. 2 for two different numbers of photons in the second field m.



a) b)

Fig. 2. Dynamics of the Schmidt parameter K obtained for two coherent states with number of photons in the second field m=10 (a) and m=20 (b) presented in the case of initial atomic state$\frac{\varphi_1+\varphi_2}{\sqrt{2}}$andinitial mean photon number$< N >=\ 10$for each field.

Since the Schmidt parameter is found for the case of two coherent field impact, its evolution partially reveals the effects of "collapse" and "revival" with regions of smooth behavior and fast oscillations changing each other. The Schmidt parameter is found to be not as high as it was for the Results of Fig. 1. The main interesting feature of the found behavior is that larger entanglement is observed for number of photons in the second fields larger than the mean number.At first glance this fact seemsto be counter-intuitivein some sense. However when the number of photons of the second field is observed to be far from the mean value, a wide range of entangled quasienergy states contribute to the bi-partite wave-function (17). As a result it is possible to tailor the degree of entanglement by choosing the number of photons in the second (control) field.

3.3 Conditional entanglement of two quantum fields

Similar strategy (as in the previous Section) can be used to examine the entanglement between two modes of quantum fields. This entanglement arises due to interaction of both fields with atomic system which seems to be one of the few good waysto induce coupling between independent modes of quantized field. Here we calculate a conditional entanglement between two quantum fields which is obtained under the condition that the atom is found to be in a certain atomic eigenstate. In this case the bipartite wave function of two fields when the atom is found in the "i-th" state is given simply by the following projection of the total wave function of the system on the "i-th" atomic state :

$$\psi(q_1, q_2) = < \varphi_i | \psi_{total} > \qquad (21)$$

Since the bi-partite state of two fields is pure and is described by the wave function (21) we can characterizethe degree of entanglement between two quantum fields by the Schmidt parameter which is given by:

$$K = \frac{1}{Sp((\rho_{red})^2)} \qquad (22)$$

where the elements of the corresponding reduced density matrix of one field can be calculated as following:

$$(\rho_{red})_{k_1}^{k_2} = \sum_m C_{ik_1m} C_{ik_2m}^* \, exp(-i\omega_1(k_1 - k_2)t) \qquad (23)$$

The calculated time-dependent entanglement between two coherent fields obtained under the condition when the atom is found to occupy the upper level is presented at Fig. 3 for the initial condition (16) with zero phase between atomic states.

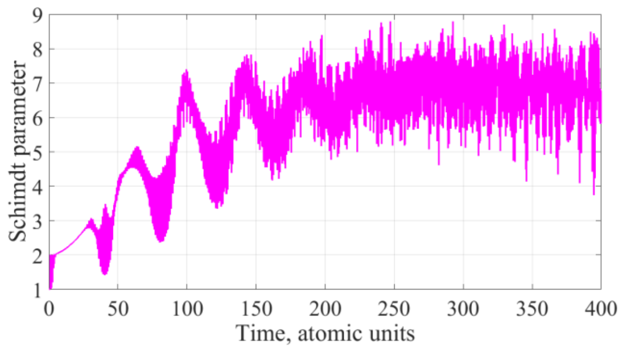
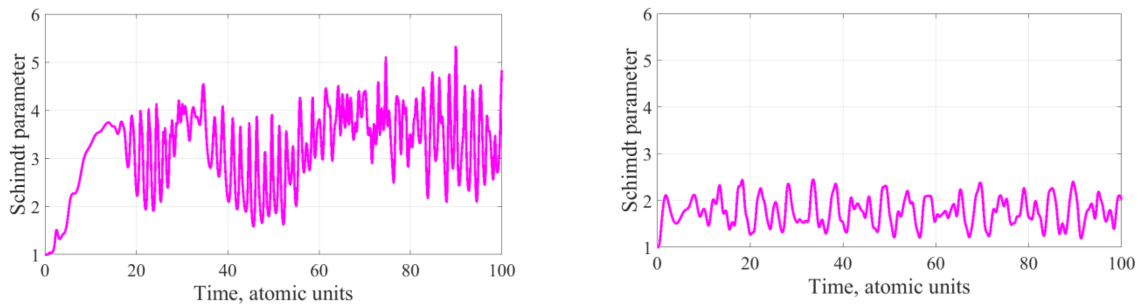


Fig. 3. The time-dependent entanglement obtained between two coherent fields of equal initial mean photon numbers $< N >= 10$under the condition when the atom is found to occupy the upper level. The initial atomic state corresponds to (16) with$\Delta = 0$.

The initial dynamics of the entanglement revials a sort of quasiperiodical behaviour with growing the magnitude sometimes accompanied by very fast oscillations. Such evolution is consistent with regimes of "collapse" and "revival" found for the populations of atomic and field states under the coherent field inpact. However the "plateau" stage on the evolution of the level probability corresponds to the smooth growing of the degree of entanglement. This fact is a clear evidence that the off-diagonal are responsible for the observed increase of entanglement and actually represent the arising of quantum correlations between fields.

At the same time the projection of the total wave function on the ground atomic state is found to provide much smaller entanglement since under used initial condition for the atom effiecient coupling between fields arises via the upper atomic level.

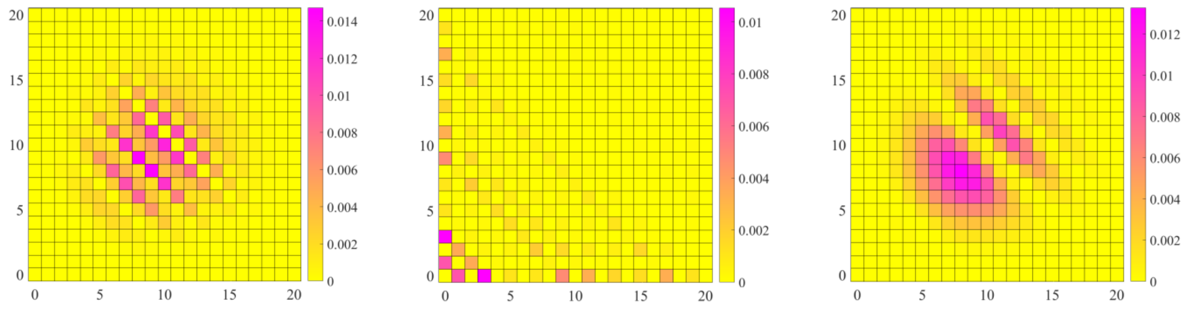
The comparison of the entanglement achieved between to coherent and two squeezed vacuum fields is presented at Fig. 4 for smaller initial mean photon number. For such small number of photons the regime of smooth growing of the entanglement degree found for coherent field states is shown to take place at the very beginning of the evolution only. However the entanglement is found to be significant and appears to be much larger in comparison to the case of two squeezed vacuum fields. The maximal value of the degree of entanglement between two squeezed vacuum fields is found to be around 2 that means a formation of a very specific bi-partite field state with a certain type of correlations between photons. To analyze the features of the bi-partite field stateы formed during the interaction with the atom we calculate the 2D photon number distribution of two fields under the condition that the atom is found to occupy the upper level: $W_{km}(t) = |C_{3km}|^2$. The data is presented at Fig. 5. for two initially coherent and two squeezed field states. Such destribution represents in particular the probability to register k photons in the first field when m photon in the second field are found. Thus, in addition to the averaged characterisitcs it gives the conditional data i.e the information about photon correlations.



a) b)

Fig. 4. Evolution of entanglement obtained between two coherent fields (a) and two squeezed vacuum fields (b) of equal initial mean photon numbers$< N >= 3$under the condition when the atom is found to occupy the upper level. The initial atomic state corresponds to (16) with$\Delta = 0$.

The distribution (a) of Fig. 5 is obtained for two initially coherent fields for instant of time in the middle of the first "plateau" on the time-dependent population of the upper atomic level. Initially both coherent fields are independent and their distribution can be simply caracterized by a circle centered at mean photon number equal to 10. During the interaction with the atom the bipartite filed distribution is characterized by well pronounced photon numbers either larger or smaller than 10.



a) b) c)

Fig. 5. The bipartite photon number distribution of two coherent (a) and squeezed vacuum (b) fields under the condition when the atom is found to occupy the upper level obtained for instant of time in the middle of the first "plateau" on the time-dependent population of the upper atomic level. (c) – the same as (a) but for the instant of time at the end of the same "plateau".

The observed very special distribution is formed due to the interaction with the atom and results from strong correlations between photons of two fields induced by this interaction. It can be seen that each photon number of the first field there is a certain predominant number of photons of the second field. This fact is a clear evidence of correlations between photons in the two fields.

In the case of two squeezed vacuum fields the photon correlationsare found to exist too but are characterized by complitely defferent features. The 2D distribution formed during the interaction with atomic system reminds a chessboard with very pronounced formation of a set of the so called N00N- states. It is the formation of such states that is responsible for the value of the entanglement degree close to 2 demonstrated at Fig. 4. It should be also noticed that such N00N- states are mostly characterized by odd numbers N since the population of the odd Fock states is much more pronounced at the 2D bipartite photon distribution for squeezed fields (Fig. 5b).

It was also found that in the regime of "collapse" of atomic transitions when the populations of allatomic and field eigenstates remain constant, the shape of 2D bipartite photon distribution and therefore photon correlations are changing dramatically. This fact is illustrated by the bi-partite distribution obtained for two initially coherent fields and calculated for another instant of time - at the end of the same "plateau"just before the second revival (See Fig. 5c). It is seen that the distribution becomes rather close to the initial one wich for two coherent fields is caracterized by a circle centered at mean photon number equal to 10.

Thus, the obtained results demonstrate the possibility to obtain new properties of bi-partite state of two modes of quantum fields and to manage the degree of their entanglementand correlations of photons not only quantitatively but also qualitatively.

3.4 General case of entanglement between two quantum fields

The results discussed above concern the case when two entangled subsystems are found in a pure state which is characterized by a wave function. For more general situation it can be not the case. If the coupled subsystems are in a mixed state and are described by a density matrix the Schmidt parameter is known not to be able to characterize their entanglement correctly. One possible way to solve this problem is to use the von Nuemann's Entropy to characterize the purity or how mixed is the studied bi-partite state and to calculate mutual Information to find the degree of entanglement or correlations [21].To perform such analysis we calculate use the von Nuemann's Entropy by following way:

$$S(\rho) = -Tr(\rho\, log_2\, \rho) = -\sum_i \lambda_i\, log_2\, \lambda_i \tag{24}$$

where density matrixρ describes the bi-partite state of two quantum fields averaged over the atomic variables.

Here λ_i are the eigenvalues of this matrix used to calculate correctly the logarithmic function of matrix andthe density matrix elements are given by:

$$\rho_{k_1m_1}^{k_2m_2} = \sum_n C_{nk_1m_1} C_{nk_2m_2}^{*}\, exp\left(-\frac{i}{\hbar}(E_{k_1m_1} - E_{k_2m_2})t\right) \tag{25}$$

The correspondent reduced matrix of one and another fields are characterized by the following matrix elements:

$$(\rho_{red\,1})_{k_1}^{k_2} = \sum_m \rho_{k_1 m}^{k_2 m}, \qquad (\rho_{red\,2})_{m_1}^{m_2} = \sum_k \rho_{k m_1}^{k m_2} \tag{26}$$

And the mutual Information can be found using the von Nuemann's Entropy calculated for the bi-partite state of two fields and the reduced density matrix for each field:

$$I_{12} = S(\rho_{red\,1}) + S(\rho_{red\,2}) - S(\rho) \tag{27}$$

The time-dependence of the Entropy obtained for the bi-partite state of two fields and for each field separately as well as the mutual Information are presented at Fig. 6 for two initially equal coherent fields with mean number of photons $< N > \ = 5$and initial atomic state chosen as$\phi_{at}(t = 0) = \frac{\varphi_1 + \varphi_2}{\sqrt{2}}$.

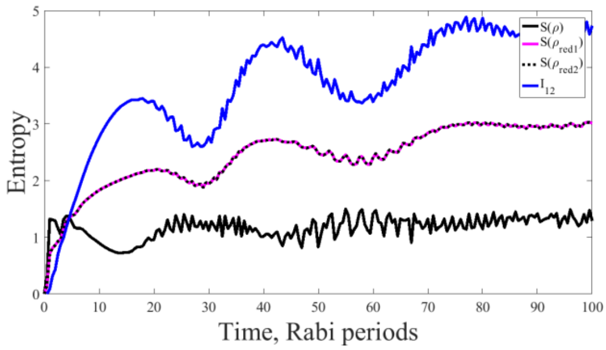


Fig. 6. The time-dependence of the Entropy obtained for the bi-partite state of two fields and for each field separately as well as the mutual Information calculated for two initially equal coherent fields with mean number of photons $< N > \ = 5$ and initial atomic state chosen as $\phi_{at}(t = 0) = \frac{\varphi_1 + \varphi_2}{\sqrt{2}}$.

For such initial condition for the atom and initially equal fields the Entropies of both fields are found to be the same. Initially all the Entropies as well as mutual Information are equal to zero since two fields are independent from each other and each field is in a pure state. However due to the interaction with the atom either bi-partite two field state or individual state of each field becomes mixed and significant correlations between the field subsystems are found. This fact is clearly demonstrated by growing with time of all the Entropies and the Information at Fig. 6. It is clearly seen thatdue to strong photon correlations an individual state of each field appears to be mixed in more extent than the bi-partite state.

Another interesting result concerns the dependence of the found photon correlations on the relative phase between acting fields. At Fig. 7 the mutual information characterizing the entanglement between fields is presented for two initially coherent fields in dependence on the phase θ of one field (see Eq. 7) and zero phase of the second field.It is seen that the phase dependence is well pronounced and the maximal entanglement isfound to take place for$\theta = 0$.

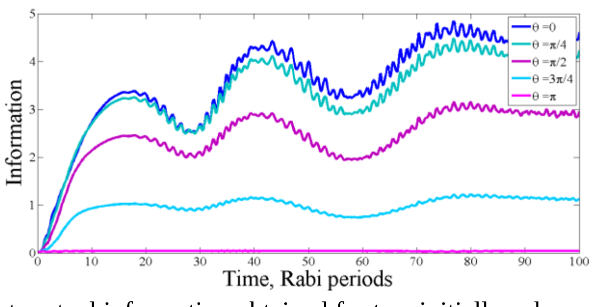
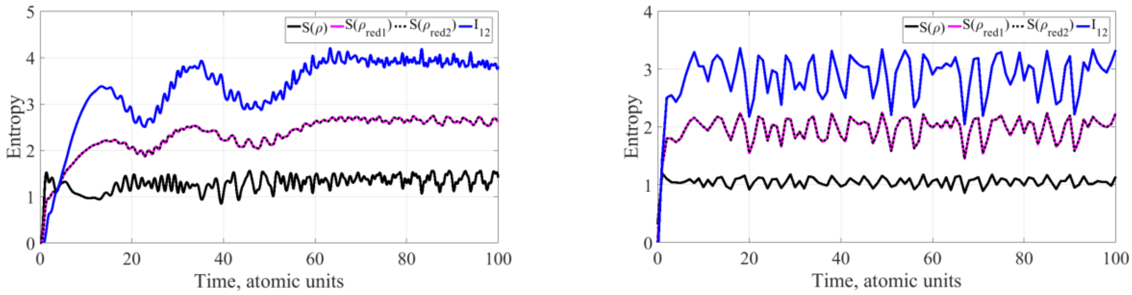


Fig. 7 The time-dependent mutual information obtained for two initially coherent fields and presented for different relative phase θbetween them.

Further we compare the entanglement achieved for initially coherent and squeezed vacuum states of fields. The results are presented at Fig. 8 for rather small mean number of photons $< N > \ = 3$. A much more regular time-dependence of the mutual information and higher reached degree of entanglement is found in the case of two initially coherent fields. However some specific mixed bi-partite state is seen to be formed for squeezed fields which is characterized by the Entropy being approximately equal to 1. This fact is due to the formation of a set of N00N- states mentioned in the previous section. The obtained results show the possibility to manage not only the entanglement between two quantum fields but also to control and tailor the bi-partite and individual states of these fields.



a) b)

Fig. 8. The time-dependence of the Entropy obtained for the bi-partite state of two fields and for each field separately as well as the mutual Information calculated for two initially equal coherent fields (a) and two initially squeezed vacuum states (b) with mean number of photons $< N > \ = 3$ and initial atomic state chosen as$\phi_{at}(t = 0) = \frac{\varphi_1+\varphi_2}{\sqrt{2}}$.

IV. CONCLUSION

The problem of interaction between three-level atomic system and two resonant electromagnetic non-classical fields is investigated and an analytical solution is obtained. The dynamics of the multipartite state is analyzed and dramatic difference is found for the cases of initially coherent and squeezed vacuum field states.

The entanglement arising between different parts of the multi-partite system is examined and ways of its quantitative description are discussed. Significantly strong entanglement between initially independent field modes is found to be induced due to their interaction via the atomic subsystem. The possibility to manage the entanglement between different interacting subsystems as well as to control and tailor the multi-partite state of the system is demonstrated. The considered system can be used as a controlled transmission unit with multi-partite entanglement resource and can be very promising for many applications in information technologies.

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