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Some New Inequalities of Lommel Wright K-Function

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Abstract
The purpose of this article is to extended lommel wright k-function and establish a new inequalities of this function. Also we have proved some new Pólya–Szegő and chebyshev type inequality. The outcome of this paper also provides a lot of Pólya–Szegő and chebyshev type inequality for several well known fractional integral operator via parameter substitutions.
Keywords: *Riemann liouville fractional integral,Pólya–Szegő inequality,chebyshev inequality,gruss type inequality, generalized lommel wright k-function.*

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I. INTRODUCTION

Fractional Calculus is an essential research area, which is equally useful not only in pure Mathematics but also in applied mathematics,physics,biology, engineering,economics and control theory etc. In recent years integral and derivative operator of fractional order are simple and important tools to generalize the classical theories and well known problem related to integer order derivative and integrals. These days, fractional integral/derivative operator are very frequently considered by the researchers working on Mathematical inequalities to extend the classical literature. one can see the well known inequalities related to integer order derivatives and integrals have been extended to fractional order derivative and integrals, These includes the inqualities of chebyshev[4], Hadmard[5], Pólya–Szegő[6], Gruss[25], Ostrowski[7], and many others.

The chebyshev inequality provide the comparision of integral mean of product of two positive function of same monotonicity to the product of their integral means.After chebyshev inequality,people start to analyze the error bounds of this inequality. For instance the well -known Gruss inequality gives the error bounds of difference of terms of the chebyshev inequality(which is well-known as chebyshev functional). the well-known Pólya–Szegő inequality gives the estimation of quotient in terms of the chebyshev inequality for bounded function. these inquelities have been studied for Riemann-Liouville and other fractional integral operators in [9-15].

Next for the result of this paper.First we give chebyshev functional and then the chebyshev functional inequality[4] as follows:

$$T(\mathrm{f},\mathrm{g}) = \frac{1}{\mathrm{b}-\mathrm{a}}\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f}(\tau)\mathrm{g}(\tau)\mathrm{d}(\tau) - \left(\frac{1}{\mathrm{b}-\mathrm{a}}\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f}(\tau)\mathrm{d}(\tau)\right)\left(\frac{1}{\mathrm{b}-\mathrm{a}}\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{g}(\tau)\mathrm{d}(\tau)\right) \tag{1}$$

where f and g are two positive and integrable function over the interval [a,b]. if f and g are synchronous on [a,b] then chebyshev inequality $\mathrm{T}(\mathrm{f},\mathrm{g}) \geq 0$ is obtained.

also one of the famous inequalities related to functional is the Gruss inequality[25] stated as follows:

$$|T(f,g)| \leq \frac{(U-u)(V-v)}{4} \tag{2}$$

where the positive and integrable function f and g satisfy

$$u \leq f(\tau) \leq U, v \leq g(\tau) \leq V$$

$$\tau \in [a,b] \ \& \ u, U, v, V \in R$$

Another more appealing and useful inequality which established the essential key of motivation in our study, which we can indicate as follows:

$$\frac{\int_a^b f^2(\tau)d\tau \int_a^b g^2(\tau)d\tau}{\left(\int_a^b f(\tau)g(\tau)d\tau\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{uv}{UV}} + \sqrt{\frac{UV}{uv}}\right)^2 \tag{3}$$

Dragomir and Diamond [16] introduced the following Gruss type integral inequality..

$$|T(f,g)| \leq \frac{(U-u)(V-v)}{4(b-a)^2\sqrt{UVuv}} \int_a^b f(\tau)d\tau \int_a^b g(\tau)d \tag{4}$$

Where positive and integrable function f and g satisfy.

$0 \leq f(\tau) \leq U <, 0 < v \leq g(\tau) \leq V < \tau \in [a,b]$ and $u, U, v, V \in R$

The purpose of this paper is to give some new Pólya–Szegő and chebyshev inequalities for generalized k-fractional integral operator containing generalized lommel wright k- function in it's kernal. In upcoming section, with define a new k- fractional integral operator containing generalized lommel wright k- function. In next section, we will utilized this k- fractional integral operator to obtain the Pólya–Szegő and chebyshev type inequalities.

1. Fractional Integral Operators

Fractional integral operators are very useful in Mathematical inequalities. A large number of fractional integral inequalities due to different types of fractional integral operators have been establish [9,14,17-24] and references. The first formulation of fractional operators is the Riemann-Liouville fractional integral operator define as follows.

Definition 1: Let $f \in L_1[a,b]$. Then Riemann -Liouville fractional integral of order $\sigma \in C$, $R(\sigma) > 0$ are defined by:

$$\left(\xi_{a^+}^{\sigma} f\right)(x) = \frac{1}{\Gamma(\sigma)}\int_a^x (x-\tau)^{\sigma-1} f(\tau)d\tau \qquad x > a \tag{5}$$

$$(\xi_{b^-}^{\sigma} f)(x) = \frac{1}{\Gamma(\sigma)}\int_x^b (\tau - x)^{\sigma-1} f(\tau)d\tau \qquad x < b \tag{6}$$

Where $\Gamma(a)$ is the gamma function defined as: $\Gamma(a) = \int_0^\infty \tau^{\sigma-1}e^{-\tau}d\tau$.

In [1] Andric et al. introduced the generalized fractional integral operators as follows:

Definition 2: Let $\varphi, \theta, \gamma \in c$, $R(\theta), R(\gamma) > 0$, $R(m) > R(\mu) > 0$, $\mu \geq 0$, $z \in C(-\infty, 0]$

Let $f \in L_1[a,b]$ and $x \in [a,b]$ then the generalized fractional integral operators are defined by:

$$\left(\xi_{\vartheta,\lambda,\varphi,a^+}^{\mu,m} f\right)(x) = \frac{1}{\Gamma(\sigma)}\int_a^x (x-\tau)^{\sigma-1} J_{\vartheta,\lambda}^{\mu,m}\left(\varphi(x-\tau)^{\theta}\right) f(\tau)d\tau \tag{7}$$

$$\left(\xi^{\mu,m}_{\vartheta,\lambda,\varphi,b^-}f\right)(x)=\frac{1}{\Gamma(\sigma)}\int_x^b(\tau-x)^{\sigma-1}J^{\mu,m}_{\vartheta,\lambda}\left(\varphi(\tau-x)^{\theta}\right)f(\tau)d\tau \tag{8}$$

Where $J^{\mu,m}_{\vartheta,\lambda}(z)$ is generalized lommel wright k- function defined by.

$$J^{\mu,m}_{\vartheta,\lambda}=\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(\lambda+n+1)^m\Gamma(\vartheta+n\mu+\lambda+1)}\left(\frac{z}{2}\right)^{2n+\vartheta+2\lambda} \tag{9}$$

$$(z\in C\,(-\infty,\,0]\,,\,m\in N,\vartheta,\lambda\in C\;\mu>0)$$

In [2], Farid introduced the unified integral operator as follows:

Definition 3: Let $f,\alpha:[a,b]\to R$, $0<a<b$ be the function such that f be a positive and integrable and α be a differentiable and strictly increasing. also $\frac{\upsilon}{x}$ be an increasing function on $[a,\infty)$ and $\sigma,\ m,\ \mu\in c,\ \ \theta\geq 0$ then for $x\in[a,b]$ the integral operators are defined by:

$$({}_{\alpha}\xi^{\upsilon;\mu,m}_{\vartheta,\lambda,\varphi,a^+}f)(x)=\int_a^x\frac{\upsilon\big(\alpha(x)-\alpha(\tau)\big)}{\alpha(x)-\alpha(\tau)}J^{\mu,m}_{\vartheta,\lambda}\left(\varphi\big(\alpha(x)-\alpha(\tau)\big)^{\theta}\right)f(\tau)d\big(\alpha(\tau)\big), \tag{10}$$

$$({}_{\alpha}\xi^{\upsilon;\mu,m}_{\vartheta,\lambda,\varphi,b^-}f(x)=\int_x^b\frac{\upsilon\big(\alpha(x)-\alpha(\tau)\big)}{\alpha(x)-\alpha(\tau)}J^{\mu,m}_{\vartheta,\lambda}\left(\varphi\big(\alpha(x)-\alpha(\tau)\big)^{\theta}\right)f(\tau)d\big(\alpha(\tau)\big) \tag{11}$$

The following generalized integral operators involving lommel wright k- function with some modification is produced, for $\phi(x)=x^{\frac{\sigma}{k}}$ with $k>0$ in [7]and [25].

Definition 4: Let $f,\alpha:[a,b]\to R$, $0<a<b$ be the function such that f be a positive and integrable and α be a differentiable and strictly increasing.let $\sigma,\ m,\ \mu\ \in\ C,\ \theta\geq\ 0$$ then for $x\in[a,b]$, $k>0$ the integral operators are defined by:

$$({}^{k}_{\alpha}\xi^{\mu,m}_{\vartheta,\lambda,\varphi,a^-}f)(x)=\int_a^x(\alpha(x)-\alpha(\tau)^{\left(\frac{\sigma}{k}-1\right)}J^{\psi,m}_{\aleph,\hbar,k}(\varphi\big(\alpha(x)-\alpha(\tau)\big)^{\frac{\theta}{k}}f(\tau)d\big(\alpha(\tau)\big) \tag{12}$$

$$({}^{k}_{\alpha}\xi^{\mu,m}_{\vartheta,\lambda,\varphi,b^-}f)(x)=\int_x^b(\alpha(x)-\alpha(\tau)^{\left(\frac{\sigma}{k}-1\right)}J^{\psi,m}_{\aleph,\hbar,k}\,(\varphi\big(\alpha(x)-\alpha(\tau)\big)^{\frac{\theta}{k}}f(\tau)d\big(\alpha(\tau)\big) \tag{13}$$

Where$J^{\psi,m}_{\aleph,\hbar,k}(z)$ is generalized lommel wright k- function defined by.

$$J^{\psi,m}_{\aleph,\hbar,k}=\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma_k(\hbar+k+nk)^m\Gamma_k(\aleph+\hbar+n\psi+k)}\left(\frac{z}{2}\right)^{2n+\frac{\aleph+2\hbar}{k}} \tag{14}$$

Remark 1:

1.The following integral operator can be deduced from (8) and (9) 1.The following integral operator produced for $\alpha(x)=x$ in (8)

$$({}^{k}\xi^{\mu,m}_{\vartheta,\lambda,\varphi,a^+})f(x)=\int_a^x(x-\tau)^{\frac{\sigma}{k}-1}J^{\psi,m}_{\aleph,\hbar,k}\left(\varphi(x-\tau)^{\frac{\theta}{k}}\right)f(\tau)d\tau \tag{15}$$

2.The following generalized Hadamard integral operator is produced for $\alpha(x)=\log(x)$ in (8).

$$({}^{k}\xi^{\mu,m}_{\vartheta,\lambda,\varphi,a^+}f(x)\;=\int_a^x\left(\left\{\log\frac{x}{\tau}\right)\right\}^{\frac{\sigma}{k}-1}J^{\psi,m}_{\aleph,\hbar,k}\left(\varphi\left(\log\frac{x}{\tau}\right)^{\frac{\theta}{k}}\right)f(\tau)\frac{d\tau}{\tau} \tag{16}$$

3.The following generalized katugampola integral operator is produced, for $\alpha(x) = \frac{x^{\rho}}{\rho}$, $\rho > 0$ in (8).

$$\left(\xi^{\mu,m}_{\vartheta,\lambda,\varphi,a^+}f\right)(x) = \int_a^x \left(\frac{x^{\rho}-\tau^{\rho}}{\rho}\right)^{\frac{\sigma}{k}-1} J^{\psi,m}_{\aleph,\hbar,k}\left(\varphi\left(\frac{x^{\rho}-\tau^{\rho}}{\rho}\right)^{\frac{\theta}{k}}\right) f(\tau)\tau^{\rho-1}d\tau \tag{17}$$

4.The following generalized (k,n) integral operator is produced, for $\alpha(x) = \frac{x^{n+1}}{n+1}$, n+1 > 0 in (8).

$$({}^{k}_{n}\xi^{\mu,m}_{\vartheta,\lambda,\varphi,a^+}f(x) = \int_a^x \left(\frac{x^{n+1}-\tau^{n+1}}{n+1}\right)^{\frac{\sigma}{k}-1} J^{\psi,m}_{\aleph,\hbar,k}\left(\varphi\left(\frac{x^{n+1}-\tau^{n+1}}{n+1}\right)^{\frac{\theta}{k}}\right) f(\tau)\tau^{n}d\tau \tag{18}$$

5.The following generalized conformable k integral operator is produced, for $\alpha(x) = \frac{x^{r+t}}{r+t}$ in (8).

$$({}^{k}_{r,t}\xi^{\mu,m}_{\vartheta,\lambda,\varphi,a^+}f(x) = \int_a^x \left(\frac{x^{r+1}-\tau^{r+1}}{r+1}\right)^{\frac{\sigma}{k}-1} J^{\psi,m}_{\aleph,\hbar,k}\left(\varphi\left(\frac{x^{r+1}-\tau^{r+1}}{r+1}\right)^{\frac{\theta}{k}}\right) f(\tau)\tau^{r}d\tau \tag{19}$$

6.The following generalized conformable (k,n) integral operator is produced for $\alpha(x) = \frac{(x-a)^n}{n}$, n > 0 in (8) and $\alpha(x) = \frac{-(b-x)^n}{n}$, n>0 in (8).

$$\left({}^{k}_{\cdot n}\xi^{\mu,m}_{\vartheta,\lambda,\varphi,a^+}f\right)(x) = \int_a^x \left(\frac{(x-a)^n-(\tau-a)^n}{n}\right)^{\frac{\sigma}{k}-1} J^{\psi,m}_{\aleph,\hbar,k}\left(\varphi\left(\frac{(x-a)^n-(\tau-a)^n}{n}\right)^{\frac{\theta}{k}}\right) f(\tau)(\tau-a)^{n-1}d\tau \tag{20}$$

$$\left({}^{k}_{\cdot n}\xi^{\mu,m}_{\vartheta,\lambda,\varphi,a^+}f\right)(x) = \int_x^b \left(\frac{(b-x)^n-(b-\tau)^n}{n}\right)^{\frac{\sigma}{k}-1} J^{\psi,m}_{\aleph,\hbar,k}\left(\varphi\left(\frac{(b-x)^n-(b-\tau)^n}{n}\right)^{\frac{\theta}{k}}\right) f(\tau)(b-\tau)^{n-1}d\tau \tag{21}$$

Remark 2: For different choice of parameter involving in the generalized lommel wright k - function (10), one can obtain new generalized integral operator.

3. Pólya Szegö and Chebyshev Type Inequalities for Generalized k - Fractional Integral Operators

In this section we obtain polya szego and chebyshev type ineqalities for generalized k-fractional integral operators containing Lommel Wright k- function in their kernels.For the reader's convenience we will use a simplified notation:

$${}^{k}_{\alpha}A_{\sigma}f(x) = \left(\xi^{\mu,m}_{\vartheta,\lambda,\varphi,a^+}f\right)(x) = \int_a^x (\alpha(x)-\alpha(\tau))^{\left(\frac{\sigma}{k}-1\right)} J^{\psi,m}_{\aleph,\hbar,k}\left(\varphi\big(\alpha(x)-\alpha(\tau)\big)^{\frac{\theta}{k}} f(\tau)d\big(\alpha(\tau)\big)\right. \tag{22}$$

$$J^{k}_{\sigma} = J^{\psi,m}_{\aleph,\hbar,k} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma_k(\hbar+k+nk)^m \Gamma_k(\aleph+\hbar+n\psi+k)} \left(\frac{z}{2}\right)^{2n+\frac{\aleph+2\hbar}{k}} \tag{23}$$

$$z \in C\,(-\infty,0],\ m \in N,\ \vartheta,\lambda \in C,\ \mu > 0$$

Theorem-1 : Suppose that (a) J_1 and J_2be two positive and integrable function on $[0,\infty)$;

(b) $\alpha'[a,b] \rightarrow R$ be an increasing and differentiable function with $\alpha' \in [a,b]$;

there exist four positive integrable function ζ_1, ζ_2, η_1 and η_2 such that

$0 < \varsigma_1(\tau) \le J_1(\tau) \le \varsigma_2(\tau), 0 < \eta(\tau) \le J_2(\tau) \le \eta(\tau),\ \tau \in [a,x], x > a$

Then for generalized k-fractional integral operator containing Lommel Wright k- function, we have

$$\frac{({}_{\alpha}^{k} A_{\sigma}\zeta_1\zeta_2 {J_2}^2)(x)({}_{\alpha}^{k} A_{\sigma}\eta_1\eta_2 {J_1}^2)(x)}{[({}_{\alpha}^{k} A_{\sigma}(\zeta_1\eta_1+\zeta_2\eta_2)J_1J_2)(x)]^2} \le \frac{1}{4} \tag{24}$$

proof: From (20) for $\tau \in [a,x]$ with $x > a$, we can write

$$\left(\frac{\zeta_2(\tau)}{\eta_1(\tau)} - \frac{J_1(\tau)}{J_2(\tau)}\right)\left(\frac{J_1(\tau)}{J_2(\tau)} - \frac{\zeta_1(\tau)}{\eta_2(\tau)}\right) \ge 0, \tag{25}$$

which implise

$$\big(\zeta_1(\tau)\eta_1(\tau) + \zeta_2(\tau)\eta_2(\tau)\big)\big(J_1(\tau)J_2(\tau)\big) \ge \eta_1(\tau)\eta_2(\tau)J_1^2(\tau) + \zeta_1(\tau)\zeta_2(\tau)J_2^2(\tau) \tag{26}$$

Multiplying (20) with $\left(\alpha(x) - \alpha(\tau)^{\left\{\frac{\sigma}{k}-1\right\}}\right) J_\sigma^k\ \left(\varphi\big(\alpha(x)\ -\ \alpha(\tau)^{\theta}\big)\alpha'(\tau)\right)$ on both side and integrating, we get

$$\int_a^x (\alpha(x) - \alpha(\tau)^{\left(\frac{\sigma}{k}-1\right)} J_\sigma^k(\varphi\big(\alpha(x) - \alpha(\tau)\big)^{\theta}(\zeta_1(\tau)\eta_1(\tau) + \zeta_2(\tau)\eta_2(\tau))J_1(\tau)J_2(\tau)\, \alpha'(\tau)d\tau \ge \int_a^x (\alpha(x) - \alpha(\tau))^{\left(\frac{\sigma}{k}-1\right)} J_\sigma^k(\varphi\big(\alpha(x) - \alpha(\tau)\big)^{\theta}\eta_1(\tau)\eta_2(\tau){J_1}^2(\tau)\alpha'(\tau)d\tau + \int_a^x (\alpha(x) - \alpha(\tau))^{\left(\frac{\sigma}{k}-1\right)} J_\sigma^k\ (\varphi\big(\alpha(x) - \alpha(\tau)\big)^{\theta}\zeta_1(\tau)\zeta_2(\tau){J_2}^2(\tau)\alpha'(\tau)d\tau \tag{27}$$

Now by using k-fractional integral operator, we get

$$({}_{\alpha}^{k}A_{\sigma}(\zeta_1\eta_1 + \zeta_2\eta_2)J_1J_2(x) \ge ({}_{\alpha}^{k}A_{\sigma}\zeta_1\zeta_2J_2^2(x))\ +\ ({}_{\alpha}^{k}A_{\sigma}\ \eta_1\eta_2\ J_1^2(x)) \tag{28}$$

By applying AM-GM inequality, we get

$$({}_{\alpha}^{k}A_{\sigma}(\zeta_1\eta_1 + \zeta_2\eta_2)J_1J_2)(x) \ge 2\sqrt{\left({}_{\alpha}^{k}A_{\sigma}\zeta_1\zeta_2J_2^2\right)(x)\left({}_{\alpha}^{k}A_{\sigma}\eta_1\eta_2J_1^2\right)(x)} \tag{29}$$

which leads to the required inequality(19).

COROLLARY 1.1

If $\zeta_1 = u, \zeta_2 = U, \eta_1 = v$ and $\eta_2 = V$, then we have

$$\frac{\left({}_{\alpha}^{k}A_{\sigma}J_1^2\right)(x)\left({}_{\alpha}^{k}A_{\sigma}J_2^2\right)(x)}{\left[\left({}_{\alpha}^{k}A_{\sigma}J_1J_2\right)(x)\right]^2} \le \frac{1}{4}\left(\sqrt{\frac{uv}{UV}} + \frac{UV}{uv}\right)^2 \tag{30}$$

THEOREM-2 : Under the assumptions of Theorem 1 with $\zeta > 0$, we have

$$\frac{({}_{\alpha}^{k}A_{\sigma}J_1^2)(x)({}_{\alpha}^{k}A_{\sigma}J_2^2)(x)({}_{\alpha}^{k}A_{\sigma}\zeta_1J_1)(x).({}_{\alpha}^{k}A_{\sigma}\eta_1J_2)(x) + ({}_{\alpha}^{k}A_{\sigma}\zeta_2J_1)({}_{\alpha}^{k}A_{\sigma}\eta_2J_2)(x)}{({}_{\alpha}^{k}A_{\sigma}\zeta_1\zeta_2)(x)({}_{\alpha}^{k}A_{\sigma}\eta_1\eta_2)} \le \frac{1}{4} \tag{31}$$

Proof. From (218), for $(\tau, k \in [a,x])$ with $(x > a)$, we can write

$$\left(\frac{\zeta_1(\tau)}{\eta_2(k)}+\frac{\zeta_2(\tau)}{\eta_1(k)}\right)\frac{J_1(\tau)}{J_2(\tau)}\geq\frac{J_1^2(\tau)}{J_2^2(\tau)}+\frac{\zeta_1(\tau)\zeta_2(\tau)}{\eta_1(k)\eta_2(k)} \tag{32}$$

which implies

$$\zeta_1(\tau)J_1(\tau)\eta_1(k)J_2(k)+\zeta_2(\tau)J_1(\tau)\eta_2(k)J_2(k)\geq\eta_1(k)\eta_2(k)J_1^2(\tau)+\zeta_1(\tau)\zeta_2(\tau)J_2^2(k) \tag{33}$$

Multiplying (22) with $(\alpha(x)-\alpha(\tau))^{\left(\frac{\sigma}{k}-1\right)}(\alpha(x)-\alpha(k))^{\frac{c}{k}-1}\ J_\sigma^k\left(\varphi(\alpha(x)-\alpha(\tau)^\theta\right)J_c^k\left(\varphi\big(\alpha(x)-\alpha(k)^\theta\big)\alpha'(\tau)\alpha'(k)\right)$ on both side and integrating, we get

$$\int_a^x\int_a^x(\alpha(x)-\alpha(\tau))^{\left(\frac{\sigma}{k}-1\right)}(\alpha(x)-\alpha(k))^{\left(\frac{c}{k}-1\right)}J_\sigma^k\left(\varphi\big(\alpha(x)-\alpha(\tau)\big)^\theta\right)J_c^k\left(\varphi\big(\alpha(x)-\alpha(k)\big)^\theta\right)\zeta_1(\tau)J_1(\tau)\eta_1(k)J_1(k)\alpha'(k)\alpha'(\tau)\ d\tau dk+\int_a^x\int_a^x(\alpha(x)-\alpha(\tau))^{\left(\frac{\sigma}{k}-1\right)}(\alpha(x)-\alpha(\tau))^{\left(\frac{c}{k}-1\right)}J_\sigma^k\left(\varphi\big(\alpha(x)-\alpha(\tau)\big)^\theta\right)J_c^k\left(\varphi\big(\alpha(x)-\alpha(k)\big)^\theta\right)\zeta_2(\tau)J_1(\tau)\eta_2(k)J_2(k)\alpha'(k)\alpha'(\tau)\ d\tau dk\geq\int_a^x\int_a^x(\alpha(x)-\alpha(\tau))^{\left(\frac{\sigma}{k}-1\right)}(\alpha(x)-\alpha(k))^{\left(\frac{c}{k}-1\right)}J_\sigma^k\left(\varphi\big(\alpha(x)-\alpha(\tau)\big)^\theta\right)J_c^k\left(\varphi\big(\alpha(x)-\alpha(k)\big)^\theta\right)\zeta_1(\tau)\zeta_2(\tau)J_2^2(k)\alpha'(k)\alpha'(\tau)\ d\tau dk+\int_a^x\int_a^x(\alpha(x)-\alpha(\tau))^{\left(\frac{\sigma}{k}-1\right)}(\alpha(x)-\alpha(k))^{\left(\frac{c}{k}-1\right)}J_\sigma^k\left(\varphi\big(\alpha(x)-\alpha(\tau)\big)^\theta\right)J_c^k\left(\varphi\big(\alpha(x)-\alpha(k)\big)^\theta\right)\eta_1(k)\eta_2(k)J_1^2(\tau)\alpha'(k)\alpha'(\tau)\ d\tau dk$$
(34)

Now by using k-fractional integral operator, we get

$$({}_\alpha^kA_\sigma\zeta_1J_1)(x)({}_\alpha^kA_c\eta_1J_2)(x)+({}_\alpha^kA_\sigma\zeta_2J_1)(x)({}_\alpha^kA_c\eta_2J_2)(x)\geq({}_\alpha^kA_\sigma J_1^2)(x)({}_\alpha^kA_c\eta_1\eta_2)(x)+({}_\alpha^kA_c\zeta_1\zeta_2)(x)({}_\alpha^kA_\sigma\zeta_1\zeta_2)(x) \tag{35}$$

By applying AM-GM inequality, we get

$$({}_\alpha^kA_\sigma\zeta_1J_1)(x)({}_\alpha^kA_c\eta_1J_2)(x)+({}_\alpha^kA_\sigma\zeta_2J_1)(x)({}_\alpha^kA_c\eta_2J_2)(x)\geq2\sqrt{({}_\alpha^kA_\sigma J_1^2)(x)({}_\alpha^kA_c\eta_1\eta_2)(x)({}_\alpha^kA_cJ_2^2)(x)({}_\alpha^kA_\sigma\zeta_1\zeta_2)(x)} \tag{36}$$

which leads to the required inequality (21).

corollary 2.1

If $\zeta_1=\mathrm{u},\zeta_2=\mathrm{U},\eta_1=\mathrm{v}$ and $\eta_2=\mathrm{V}$ then we have

$$\frac{(\chi_\sigma(x)\chi_c(x)({}_\alpha^kA_\sigma J_1^2)(x)({}_\alpha^kA_cJ_2^2)(x)}{\left(({}_\alpha^kA_\sigma J_1)(x)({}_\alpha^kA_cJ_2)(x)\right)^2}\leq\frac{1}{4}\left(\sqrt{\frac{uv}{UV}}+\frac{UV}{uv}\right)^2 \tag{37}$$

<u>Theoram</u> 3

Under the assumption of Theorem 1 with $(c>0)$, we have

$$({}_\alpha^kA_\sigma J_1^2)(x)\,({}_\alpha^kA_cJ_2^2)(x)\ \leq\ ({}_\alpha^kA_\sigma(\zeta_2J_1J_2/\eta_1))(x)\,({}_\alpha^kA_\sigma(\eta_2J_1J_2/\zeta_1)\,(x) \tag{38}$$

Proof: From (18), for $\tau\in[a,x]$ with $x>a$, we can write

$$\frac{\zeta_2(\tau)J_1(\tau)J_2(\tau)}{\eta_1(\tau)}-J_1^2(\tau)\geq0 \tag{39}$$

$$\frac{\eta_2(\tau)J_1(\tau)J_2(\tau)}{\zeta_1(\tau)}-J_2^2(\tau)\geq0 \tag{40}$$

Multiply by $(\alpha(x)-\alpha(\tau))^{\left(\frac{\sigma}{k}-1\right)} J_\sigma^k\left(\varphi\left(\alpha(x)-\alpha(\tau)\right)^\theta\right)\alpha'(\tau)$ *and* $(\alpha(x)-\alpha(\tau)^{\left(\frac{c}{k}-1\right)} J_c^k\left(\varphi\left(\alpha(x)-\alpha(k)\right)^\theta\right)\alpha'(k)$ on both side and integrating. we get

$$\int_a^x(\alpha(x)-\alpha(\tau))^{\left(\frac{\sigma}{k}-1\right)} J_\sigma^k\left(\varphi\left(\alpha(x)-\alpha(\tau)\right)^\theta\right)J_1^2(\tau)\alpha'(\tau)d\tau$$
$$\leq\int_a^x(\alpha(x)-\alpha(\tau))^{\left(\frac{\sigma}{k}-1\right)} J_\sigma^k\left(\varphi\left(\alpha(x)-\alpha(\tau)\right)^\theta\right)\frac{\zeta_2(\tau)}{\eta_1(\tau)}J_1(\tau)J_2(\tau)\alpha'(\tau)d\tau \tag{41}$$

And

$$\int_a^x(\alpha(x)-\alpha(k))^{\left(\frac{c}{k}-1\right)} J_c^k\left(\varphi\left(\alpha(x)-\alpha(k)\right)^\theta\right)J_1^2(k)\alpha'(k)dk$$
$$\leq\int_a^x(\alpha(x)-\alpha(k))^{\left(\frac{c}{k}-1\right)} J_c^k\left(\varphi\left(\alpha(x)-\alpha(k)\right)^\theta\right)\frac{\eta_2(k)}{\zeta_1(k)}J_1(k)J_2(k)\alpha'(k)dk \tag{42}$$

Now by using k - fractional integral operator we get.

$$({}_\alpha^kA_\sigma J_1^2)(x) \ \leq\ ({}_\alpha^kA_\sigma(\zeta_2J_1J_2/\eta_1))(x) \tag{43}$$

and

$$({}_\alpha^kA_cJ_2^2)(x) \ \leq\ ({}_\alpha^kA_\sigma(\eta_2J_1J_2/\zeta_1))\,(x) \tag{44}$$

Multiplying (26) with (27) , we obtain (23).

Corollary 3.1

if $\zeta_1=\mathrm{u}, \zeta_2=\mathrm{U}, \eta_1=\mathrm{v}$ and $\eta_2=\mathrm{V}$ then we have

$$\frac{({}_\alpha^kA_\sigma J_1^2)(x)\ ({}_\alpha^kA_cJ_2^2)(x)}{({}_\alpha^kA_\sigma J_1(x)J_2(x))\ ({}_\alpha^kA_cJ_1(x)J_2(x))}\leq\frac{UV}{uv} \tag{45}$$

The chebyshev type inequalities for generalized k-fractional integral operators given as follows:

Theorem 4

Under the assumptions of Theorem 1 with $(c>0)$, we have

$$|\chi_\sigma(x)({}_\alpha^kA_\sigma J_1J_2)(x)+\chi_c(x)({}_\alpha^kA_\sigma J_1J_2)(x)-({}_\alpha^kA_\sigma J_1)(x)({}_\alpha^kA_cJ_2)(x)-({}_\alpha^kA_\sigma J_2)(x)({}_\alpha^kA_cJ_1)(x)|$$
$$\leq\left|G_{\sigma,c}(J_1,\zeta_1,\zeta_2(x)+G_{c,\sigma}(J_2,\zeta_1,\zeta_2(x)\right|^{\frac{1}{2}}\times\left|G_{\sigma,c}(J_1,\eta_1,\eta_2(x)+G_{c,\sigma}(J_2,\eta_1,\eta_2(x)\right|^{\frac{1}{2}} \tag{46}$$

Proof: Let f and g be two positive and integrable functions on $([0,\infty))$. For $(\tau,\mathrm{k}\in[a,x])$ with $(\mathrm{x}>\mathrm{a})$, we define $\mathrm{L}(\tau,\mathrm{k})$ as

$$\mathrm{L}(\tau,\mathrm{k})\ =\ (J_1(\tau)-J_1(\mathrm{k})(J_2(\tau)-J_2(\mathrm{k}) \tag{47}$$

which implies

$$\mathrm{L}(\tau,\mathrm{k})\ =\ J_1(\tau)J_2(\tau)+J_1(\mathrm{k})J_2(\mathrm{k})-J_1(\tau)J_2(\mathrm{k})-J_1(\mathrm{k})J_2(\tau) \tag{48}$$

Multiplying (30) by $(\alpha(x)-\alpha(\tau))^{\left(\frac{\sigma}{k}-1\right)}(\alpha(x)-\alpha(k))^{\left(\frac{c}{k}-1\right)}J_\sigma^k\,\varphi\left(\alpha(x)-\alpha(\tau)\right)^\theta)J_c^k\left(\varphi\left(\alpha(x)-\alpha(k)\right)^\theta\right)\alpha'(k)\alpha'(\tau)$ and integrating, we get

$$\int_a^x\int_a^x(\alpha(x)-\alpha(\tau))^{\left(\frac{\sigma}{k}-1\right)}(\alpha(x)-\alpha(k))^{\left(\frac{c}{k}-1\right)}J_\sigma^k(\varphi\left(\alpha(x)-\alpha(\tau)\right)^\theta)J_c^k\left(\varphi\left(\alpha(x)-\alpha(k)\right)^\theta\right)\mathrm{L}(\tau,\mathrm{k})\alpha'(k)\alpha'(\tau)\ d\tau dk \tag{49}$$

Now by using k-fractional integral operator, we have

$$\chi_c(x)({}_k^{\alpha}A_{\sigma}J_1J_2)(x) + \chi_{\sigma}(x)({}_{\sigma}^{k}A_cJ_1J_2)(x) - ({}_{\alpha}^{x}A_{\sigma}J_1)(x)({}_{\alpha}^{k}A_cJ_2)(x) - ({}_{\alpha}^{k}A_cJ_1)(x)({}_k^{\alpha}A_{\sigma}J_2)(x) \quad (50)$$

By using Cauchy - Schwartz inequality, we have

$$\int_a^x \int_a^x (\alpha(x)-\alpha(\tau))^{\left(\frac{\sigma}{k}-1\right)} (\alpha(x)-\alpha(k))^{\left(\frac{c}{k}-1\right)} J_\sigma^k(\varphi(\alpha(x)-\alpha(\tau))^{\theta}) J_c^k\, \varphi(\alpha(x)-\alpha(k))^{\theta}) L(\tau,k)\alpha'(k)\alpha'(\tau)\; d\tau dk \le \Big[\int_a^x \int_a^x (\alpha(x)-\alpha(\tau))^{\left(\frac{\sigma}{k}-1\right)} (\alpha(x)-\alpha(k))^{\left(\frac{c}{k}-1\right)} J_\sigma^k(\varphi(\alpha(x)-\alpha(\tau))^{\theta}) J_c^k\,(\varphi(\alpha(x)-\alpha(k))^{\theta}) L(\tau,k)\alpha'(k)\alpha'(\tau) J_1^2(\tau)\, d\tau dk + \int_a^x \int_a^x (\alpha(x)-\alpha(\tau))^{\left(\frac{\sigma}{k}-1\right)} (\alpha(x)-\alpha(k))^{\left(\frac{c}{k}-1\right)} J_\sigma^k(\varphi(\alpha(x)-\alpha(\tau))^{\theta}) J_c^k\,(\varphi(\alpha(x)-\alpha(k))^{\theta}) L(\tau,k)\alpha'(k)\alpha'(\tau) J_1^2(k)\, d\tau dk - 2\int_a^x \int_a^x (\alpha(x)-\alpha(\tau))^{\left(\frac{\sigma}{k}-1\right)} (\alpha(x)-\alpha(k))^{\left(\frac{c}{k}-1\right)} J_\sigma^k(\varphi(\alpha(x)-\alpha(\tau))^{\theta}) J_c^k\,(\varphi(\alpha(x)-\alpha(k))^{\theta}) L(\tau,k)\alpha'(k)\alpha'(\tau)\, J_1(\tau) J_2\,(k) d\tau dk\Big]^{\frac{1}{2}} \le [\chi_c(x)({}_{\alpha}^{k}A_{\sigma}J_1^2)(x) + \chi_{\sigma}(x)({}_{\alpha}^{k}A_cJ_1^2)(x) - 2({}_{\alpha}^{k}A_{\sigma}J_1)(x)({}_{\alpha}^{k}A_cJ_1)(x)]^{\frac{1}{2}} \times [\chi_c(x)({}_{\alpha}^{k}A_{\sigma}J_2^2)(x) + \chi_{\sigma}(x)({}_{\alpha}^{k}A_cJ_2^2)(x) - 2({}_{\alpha}^{k}A_{\sigma}J_2)(x)({}_{\alpha}^{k}A_cJ_2)(x)]^{\frac{1}{2}} \quad (51)$$

By taking $\eta_1(t) = \eta_2(t) = J_2(t) = 1$ in theorem 1, we get following inequality

$$({}_{\alpha}^{k}A_{\sigma}J_1^2)(x) \le \frac{[(({}_{\alpha}^{k}A_{\sigma}(\zeta_1+\zeta_2)J_1)(x)]^2}{4\big(({}_{\alpha}^{k}A_{\sigma}J_1^2)(\zeta_1\zeta_2)\big)(x)} \quad (52)$$

this <u>implise</u>

$$\chi_c(x)({}_{\alpha}^{k}A_{\sigma}J_1^2)(x) - ({}_{\alpha}^{x}A_{\sigma}J_1)(x)({}_{\alpha}^{k}A_cJ_1)(x) \le \frac{[\chi_c(x)({}_{\alpha}^{x}A_{\sigma}(\zeta_1+\zeta_2)J_1)(x)]^2}{4({}_{\alpha}^{k}A_{\sigma}(\zeta_1\zeta_2))(x)} - ({}_{\alpha}^{x}A_{\sigma}J_1)(x)({}_{\alpha}^{k}A_cJ_1)(x) = G_{\sigma,c}(J_1,\zeta_1,\zeta_2)(x) \quad (53)$$

and

$$\chi_c(x)({}_{\alpha}^{x}A_{\sigma}J_1^2)(x) - ({}_{\alpha}^{x}A_{\sigma}J_1)(x)({}_{\alpha}^{k}A_cJ_1)(x) \le \frac{[\chi_c(x)({}_{\alpha}^{x}A_{\sigma}(\zeta_1+\zeta_2)J_1)(x)]^2}{4({}_{\alpha}^{k}A_{\sigma}(\zeta_1\zeta_2))(x)} - ({}_{\alpha}^{x}A_{\sigma}J_1)(x)({}_{\alpha}^{k}A_cJ_2)(x) = G_{c,\sigma}(J_1,\zeta_1,\zeta_2)(x) \quad (54)$$

Apply the same procedure for $J_2(t) = \zeta_1(t) = \zeta_2(t)$=1we get following inequality:

$$\chi_c(x)({}_{\alpha}^{x}A_{\sigma}J_2^2)(x) - ({}_{\alpha}^{k}A_{\sigma}J_2)(x)({}_{\alpha}^{k}A_cJ_2)(x) \le G_{\sigma,c}(J_2,\eta_1,\eta_2)(x) \quad (55)$$

and

$$\chi_{\sigma}(x)({}_{\alpha}^{x}A_cJ_2^2)(x) - ({}_{\alpha}^{k}A_{\sigma}J_2)(x)({}_{\alpha}^{k}A_cJ_2)(x) \le G_{c,\sigma}(2,\zeta_1,\zeta_2)(x) \quad (56)$$

Finally considering (31) to (35), we arrive at the desired result in (28).

Theorem-5

Under the assumptions of Theorem 4, we have

$$|\chi_{\sigma}(x)({}_{\alpha}^{k}A_{\sigma}J_1J_2)(x) - ({}_{\alpha}^{k}A_{\sigma}J_1)(x)({}_{\alpha}^{k}A_{\sigma}J_2)(x) \le \left|G_{\sigma,\sigma}(J_1,\zeta_1,\zeta_2)(x)G_{\sigma,\sigma}(J_2,\eta_1,\eta_2)(x)\right|^{\frac{1}{2}}. \quad (57)$$

Where $$G_{\alpha,\sigma}(m,n,o)(x) = \frac{\chi_{\sigma}(x)[({}_{\alpha}^{k}A_{\sigma}(n+0)m)(x)]^2}{4({}_{\alpha}^{k}A_{\alpha}n0)(x)} - ({}_{\alpha}^{k}A_{\sigma}m)(x)({}_{\alpha}^{k}A_{\sigma}m)(x) \quad (58)$$

Corollary-4

If $\zeta_1 = u$, $\zeta_2 = U$, $\eta_1 = v$, $\eta_2 = V$ then we have

$$|\chi_{\sigma}(x)({}_{\alpha}^{k}A_{\sigma}J_1J_2)(x) - ({}_{\alpha}^{k}A_{\sigma}J_1(x)({}_{\alpha}^{k}A_{\sigma}J_2(x)) \le \frac{(U-u)(V-v)}{4\sqrt{uUvV}}({}_{\alpha}^{k}A_{\sigma}J_1)(x)({}_{\alpha}^{k}A_{\sigma}J_2)(x) \quad (59)$$

Remark: In Th. 4 and Th.5, for $\alpha(x) = x, k = 1$,we get Th. 4, Co.4.

II. CONCLUSIONS

We have proved some new Pólya–Szegő and chebyshev type inequality for generalized k-fractional integral operator involving Lommel Wright k-function in their kernels. The outcomes of this paper also provides a lot of Pólya–Szegő and chebyshev type inequalities for several well known fractional integral operators via parameter substitutions.

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