FEKETE SZEGO Coefficient Inequality of Regular Functions for A Special Class

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ABSTRACT: We will consider new type of family of analytic functions and its subclasses will be discussed here, by which coefficient bounds of FeketeSzego functional $a_3 - \mu a_2^2$ for the analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ fitting in these classes and subclasses, will be obtained.

KEYWORDS: Univalent functions, Coefficient inequality, Starlike functions, Convex functions, Close to convex functions and bounded functions.

MATHEMATICS SUBJECT CLASSIFICATION: 30C50

I. Introduction:

Let $\mathcal{A}$ denote the family of functions of the type

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n (1.1)$$

regular in the unit disc $E = \{z | |z| < 1\}$. Let the family of functions of the form (1.1) which are analytic and univalent in $E$ be denoted by $\mathcal{S}$.

Bieber Bach ( [7], [8] ) proved in 1916, that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. Löwner [5] proved in 1923, that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$.

With the recognized estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, naturally some relation was to be sought between $a_3$ and $a_2^2$ for the class $\mathcal{S}$. Löwner’s method was used by Fekete and Szegö[9] to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1 - \mu}\right), & \text{if } 0 \leq \mu \leq 1; \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases} (1.2)$$

The inequality (1.2) plays a crucial role in determining approximations of higher order coefficients for some subclasses $\mathcal{S}$ (See Chihchra[1], Babalola[6]).

Let us outline some subclasses of $\mathcal{S}$.

We will denote by $\mathcal{S}^*$, the family of univalent and starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$$

and satisfying the condition

$$\text{Re} \left( \frac{zg'(z)}{g(z)} \right) > 0, z \in E. \quad (1.3)$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A}$$

and satisfying the condition

$$\text{Re} \left( \frac{(zh'(z))}{h(z)} \right) > 0, z \in E. \quad (1.4)$$
A function \( f(z) \in \mathcal{A} \) is known as close to convex function if there exists \( g(z) \in S^* \) such that
\[
\text{Re} \left( \frac{zf'(z)}{g(z)} \right) > 0, z \in \mathbb{E}.
\] (1.5)

Kaplan [3] familiarized us with the class of close to convex functions and denoted it by \( C \) and proved that all close to convex functions are univalent.

We introduced a new subclass
\[
\left\{ f(z) \in \mathcal{A}; \frac{z(f'(z))^2 + f(z)f''(z)}{f(z)f'(z)} < \left( \frac{1 + Az}{1 + Bz} \right) \delta \right\}; z \in \mathbb{E}
\]
and we will denote it as \( S^*(f, f', f'', A, B, \delta) \).

Symbol \(< \) stands for subordination, which we describe as follows:

**Principle of Subordination:** Let \( f(z) \) and \( F(z) \) be two functions analytic in \( \mathbb{E} \). Then \( f(z) \) is called subordinate to \( F(z) \) in \( \mathbb{E} \) if there exists a function \( w(z) \) analytic in \( \mathbb{E} \) satisfying the conditions \( w(0) = 0 \) and \( |w(z)| < 1 \) such that \( f(z) \approx F(w(z)); z \in \mathbb{E} \) and we write \( f(z) \prec F(z) \).

By \( \mathcal{U} \), we denote the class of analytic bounded functions of the form
\[
w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1.
\] (1.8)

It is known that
\[
|d_1| \leq 1, |d_2| \leq 1 - |d_1|^2.
\] (1.9)

**II. PRELIMINARY LEMMAS:**

For \( 0 < \epsilon < 1 \), we write
\[
w(z) = \left( \frac{\epsilon + z}{1 + \epsilon z} \right)
\]
so that
\[
\left( \frac{1 + Aw(z)}{1 + Bw(z)} \right) \delta = 1 + (A - B)\delta c_1 z + (A - B)\delta c_2 - B\delta c_1^2 z^2 + - - -
\] (2.1)

**III. MAIN RESULTS**

**THEOREM 3.1:** Let \( f(z) \in S^*(f, f', f'', A, B, \delta) \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{1}{72} - \frac{\delta(5A - 14B)}{8} \mu ; & \text{if } \mu \leq \frac{\delta(5A - 14B) - 9}{8\delta(A - B)} \\
\frac{\delta(5A - 14B)}{8} \mu ; & \text{if } \frac{\delta(5A - 14B) - 9}{8\delta(A - B)} \leq \mu \leq \frac{\delta(5A - 14B) + 9}{8\delta(A - B)} \\
\frac{\delta^2(5A - 14B)}{72} \mu ; & \text{if } \mu \geq \frac{\delta(5A - 14B) + 9}{8\delta(A - B)}
\end{cases}
\] (3.1)

The results are sharp.

**Proof:** By definition of \( f(z) \in S^*(A; B) \), we have
\[
z[(f'(z))^2 + f(z)f''(z)] = \left( \frac{1 + Aw(z)}{1 + Bw(z)} \right) \delta ; w(z) \in \mathcal{U}.
\] (3.4)

Expanding the series (3.4), we get
\[
\{1 + 6a_2 + (6a_2^2 + 12a_2^2)z^2 + - - - \} = [1 + [(A - B)\delta c_1 + 3a_2]z + [\delta(A - B)(c_2 - B\delta c_1^2) + 3a_2(A - B)\delta c_1 + 4a_3 + 2a_2^2]z^2 + - - - \}
\] (3.5)

Identifying terms in (3.5), we get
\[
a_2 = \frac{(A - B)\delta}{8} c_1 \quad (3.6)
a_3 = \frac{\delta(A - B)}{8} c_2 + \frac{\delta^2(5A - 14B)}{72} c_1^2 \quad (3.7)
\]
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From (3.6) and (3.7), we obtain
\[ a_3 - \mu a_2^2 = \frac{\delta(A-B)}{8} c_2 + \frac{\delta^2(A-B)}{2} \left( \frac{(5A-14B)}{72} - \frac{(A-B)}{9-\mu} \right) c_1^2. \quad (3.8) \]

Taking absolute value, (3.8) can be rewritten as
\[ |a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{8} |c_2| + \frac{\delta^2(A-B)}{2} \left( \frac{(5A-14B)}{72} - \frac{(A-B)}{9-\mu} \right) |c_1|^2. \quad (3.9) \]

Using (1.9) in (3.9), we get
\[ |a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{8} \left( 1 - |c_1|^2 \right) + \frac{\delta^2(A-B)}{2} \left( \frac{1}{3^n} - \frac{(A-B)}{2^{n-1} - \mu} \right) |c_1|^2 \]
\[ = \frac{\delta(A-B)}{8} + \left( \frac{\delta^2(A-B)(5A-14B)}{72} - \frac{\delta^2(A-B)^2}{9} - \frac{\delta^2(A-B)}{9} \right) |c_1|^2. \quad (3.10) \]

Case I: \( \mu \leq \frac{9(5A-14B)}{8(A-B)} \)
(3.10) can be rewritten as
\[ |a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{8} \left( 1 - |c_1|^2 \right) + \frac{\delta^2(A-B)[\delta(5A-14B) - 9]}{72} - \frac{\delta^2(A-B)^2}{9} \mu |c_1|^2. \quad (3.11) \]

Subcase I (a): \( \mu \leq \frac{[\delta(5A-14B) - 9]}{\delta.8(A-B)} \)
Using (1.9), (3.11) becomes
\[ |a_3 - \mu a_2^2| \leq \frac{\delta(A-B)[\delta(5A-14B) - 9]}{72} - \frac{\delta^2(A-B)^2}{9} \mu |c_1|^2. \quad (3.12) \]

Subcase I (b): \( \mu \geq \frac{[\delta(5A-14B) - 9]}{\delta.8(A-B)} \)
We obtain from (3.11)
\[ |a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{8} \quad (3.13) \]

Case II: \( \mu \geq \frac{9(5A-14B)}{8(A-B)} \)

Preceding as in case I, we get
\[ |a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{8} + \left( \frac{\delta^2(A-B)^2}{9} - \frac{\delta(A-B)[\delta(5A-14B) + 9]}{72} \right) |c_1|^2. \quad (3.14) \]

Subcase II (a): \( \mu \leq \frac{8\delta(5A-14B) + 9}{8\delta(A-B)} \)
(3.14) takes the form
\[ |a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{8} \quad (3.15) \]

Combining the results of subcases I(b) and II(a), we can write
\[ |a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{8}; \quad \text{if } \frac{[\delta(5A-14B) - 9]}{\delta.8(A-B)} \leq \mu \leq \frac{8\delta(5A-14B) + 9}{8\delta(A-B)} \quad (3.16) \]

Subcase II (b): \( \mu \geq \frac{8\delta(5A-14B) + 9}{8\delta(A-B)} \)

Preceding as in subcase I (a), we get
\[ |a_3 - \mu a_2^2| \leq \frac{\delta^2(A-B)^2}{9} - \frac{\delta^2(A-B)(5A-14B)}{72} \quad (3.17) \]

Combining (3.12), (3.16) and (3.17), the theorem is established.

Extremal function for (3.1) and (3.3) is demarcated by
\[ f_1(z) = z \left( 1 + \frac{p^2}{(p^2 - 2q)} \right)^{p^2 - 2q} \]

Extremal function for (3.2) is defined by
\[ f_2(z) = z(1 + z^2)^q \]

Where \( p = \frac{\delta(A-B)}{3} \) and \( q = \frac{(A-B)[\delta(5A-14B)]}{72} \)

Corollary 3.2: Putting \( A = 1, B = -1 \) and \( \delta = 1 \) in the theorem, we get
These approximations were derived by G. Singh [6] and are outcomes for the class of univalent functions.

References:


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