International Journal of Research in Engineering and Science (IJRES)
ISSN (Online): 2320-9364, ISSN (Print): 2320-9356
www.ijres.org Volume 10 Issue 8 ǁ August 2022 ǁ PP. 474-476

An advance subclass of Analytic Functions having a unique coefficient inequality

Gurmeet Singh
Department of Mathematics, Khalsa College, Patiala

Misha Rani
Research Fellow, PunjabiUniversity, Patiala

***Abstract** –In our present work, we use the collection of analytic functions which are of the form f (z) = z +$\sum_{k=2}^{\infty} a_k z^k$, to solve an inequality called FeketeSzegö Inequality for a new well defined class.*
***2010 Mathematics Subject Classification**: 30C45, 30C50*.
***Keywords** - Principle of subordination, Fekete – Szegö Inequality and concept of Bounded analytic functions.*

Date of Submission: 13-08-2022 Date of acceptance: 28-08-2022

I. Introduction

Fekete - Szegö Inequality is an inequality which is concerned with those coefficients that are related to univalent analytic functions [8], [16]. It was founded by M. Fekete and G. Szegö in 1933[5]. It is related to Bieberbach conjecture([6], [15], [16], [17]), which gives the necessary condition to map the unit disk of a complex plane injectively to the complex plane. This was given by L. Bieberbach [2] in 1916 but proved finally by Louis De Branges [3] in 1985.

Let us define some fundamental classes and results:-

A consists the family of functions *f* having the Taylor's expansion

$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$

with the normalization $f(0) = 0$, $f'(0) = 1$ and *f* must be analytic in $E = \{z \in C: |z| < 1\}$.

S consists the family of functions *f* normalizedby $f(0) = 0$, $f'(0) = 1$ and

$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$

which is univalent in the open disk $E = \{ z \in C: |z| < 1\}$

$S^*(\phi)$ be the class of functions in $f \in S$, for which $\frac{z f'(z)}{f(z)} \prec \phi(z)$,given by Ma and Minda [12].

Let *h*(z) be any analytic function in *E* of the form $h(z) = \sum_{n=1}^{\infty} c_n z^n$, then it is said to be Schwarzian function and the class is denoted by U, if the conditions $h(0) = 0$ and $|h(z)| < 1$ hold. The necessary conditions, given by Miller et. al. [13], are $|c_1| \le 1$, $|c_2| \le 1 - |c_1|^2$

Let us suppose that we have two analytic functions*u* (z) and *v*(z) in *E*. If there existsa bounded analytic function *F(z)*, giving the relation that $|F(z)| < 1$,$F(0) = 0$ *and* $u(z) = v(F(z))$;$z \in E$,then we will say that the function *u*(z) is subordinate to *v(z)*, written as $u(z) \prec v(z)$ and this concept (called subordination) was given by Lindelof [11].

Here, we introduce a new class $S^*(f, f', f'', fof)$ of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$; defined as

$$\alpha\left(\frac{z\left\{\left(f'(z)\right)^2 + f(z) f''(z)\right\}}{f(z) f'(z)}\right) + (1-\alpha)\frac{z f'\{f(z)\} f'(z)}{f\{f(z)\}} = \frac{1+w(z)}{1-w(z)};\ z \in E. \quad ...(1.1)$$

II. Main Results:-

THEOREM-1:- Let $f(z) \in S^*(f, f', f'', fof)$ and $\phi(z) = \frac{1+w(z)}{1-w(z)}$;*w(z)* is a bounded analytic function , then

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{2(\alpha^2+50\alpha-32)}{(4-\alpha)^2(7+\alpha)} - \dfrac{4\mu}{(4-\alpha)^2}\ ;\ \mu \le \dfrac{(29\alpha-24)}{(7+\alpha)}\ ; \\ \dfrac{2}{(7+\alpha)}\ ;\dfrac{(29\alpha-24)}{(7+\alpha)} \le \mu \le \dfrac{(\alpha^2+21\alpha-8)}{(7+\alpha)}\ ; \\ \dfrac{4\mu}{(4-\alpha)^2} - \dfrac{2(\alpha^2+50\alpha-32)}{(4-\alpha)^2(7+\alpha)}; \mu \ge \dfrac{(\alpha^2+21\alpha-8)}{(7+\alpha)}\ . \end{cases}$$

PROOF: - By definition of $S^*(f, f', f'', fof)$, given by (1.1)

and using $w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \ldots$
$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$
$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \ldots$
$f''(z) = 2a_2 + 6a_3 z + 12a_4 z^2 + \ldots$
$f\{f(z)\} = z + 2a_2 z^2 + 2a_3 z^3 + 2a_2^2 z^3 \ldots$
$f'\{f(z)\} = 1 + 4a_2 z + 6a_3 z^2 + 6a_2^2 z^2 + \ldots$
we get
$1 + (4-\alpha)a_2 z + [(7+\alpha)a_3 + (24-29\alpha)a_2^2]z^2 + \ldots = 1 + 2c_1 z + 2(c_2 + c_1^2)z^2 + \ldots$
Comparing like coefficients, one can easily obtain
$a_2 = \frac{2c_1}{4-\alpha}$ and $a_3 = \frac{2c_2}{(7+\alpha)} + \frac{2\alpha^2+100\alpha-64}{(4-\alpha)^2(7+\alpha)} c_1^2$
Using these values of a_2 and a_3, one can construct
$a_3 - \mu a_2^2 = \frac{2c_2}{(7+\alpha)} + \left(\frac{2\alpha^2+100\alpha-64}{(4-\alpha)^2(7+\alpha)} - \frac{4\mu}{(4-\alpha)^2}\right) c_1^2$
After applying mode on both sides, we get
$|a_3 - \mu a_2^2| \le \left(\frac{2}{(7+\alpha)}\right)|c_2| + \left|\frac{2\alpha^2+100\alpha-64}{(4-\alpha)^2(7+\alpha)} - \frac{4\mu}{(4-\alpha)^2}\right| |c_1|^2$
Using $|c_2| \le 1 - |c_1|^2$, we get
$|a_3 - \mu a_2^2| \le \frac{2}{(7+\alpha)} + \left\{\left|\frac{2\alpha^2+100\alpha-64}{(4-\alpha)^2(7+\alpha)} - \frac{4\mu}{(4-\alpha)^2}\right| - \frac{2}{(7+\alpha)}\right\} |c_1|^2$
Case 1:- If $\mu \le \frac{\alpha^2+50\alpha-32}{2(7+\alpha)}$
Then, $|a_3 - \mu a_2^2| \le \frac{2}{(7+\alpha)} + \left\{\frac{4(29\alpha-24)}{(4-\alpha)^2(7+\alpha)} - \frac{4\mu}{(4-\alpha)^2}\right\} |c_1|^2$
Subcase – 1 (a):- When $\mu \le \frac{(29\alpha-24)}{(7+\alpha)}$
By using $|c_1| \le 1$, we get
$|a_3 - \mu a_2^2| \le \frac{2(\alpha^2+50\alpha-32)}{(4-\alpha)^2(7+\alpha)} - \frac{4\mu}{(4-\alpha)^2}$ ---------- (1.2)
Subcase – 1 (b):- When $\mu \ge \frac{(29\alpha-24)}{(7+\alpha)}$
Then, $|a_3 - \mu a_2^2| \le \frac{1}{(\alpha+3)}$ -------- (1.3)
Case – 2:- If $\mu \ge \frac{\alpha^2+50\alpha-32}{2(7+\alpha)}$
Then, $|a_3 - \mu a_2^2| \le \frac{2}{(7+\alpha)} + \left\{\frac{4\mu}{(4-\alpha)^2} - \frac{4(\alpha^2+21\alpha-8)}{(4-\alpha)^2(7+\alpha)}\right\} |c_1|^2$
Subcase-2 (a):- When $\mu \ge \frac{(\alpha^2+21\alpha-8)}{(7+\alpha)}$
Then, $|a_3 - \mu a_2^2| \le \frac{4\mu}{(4-\alpha)^2} - \frac{2(\alpha^2+50\alpha-32)}{(4-\alpha)^2(7+\alpha)}$ ----- (1.4)
Subcase – 2 (b):- When $\mu \le \frac{(\alpha^2+21\alpha-8)}{(7+\alpha)}$
Then, $|a_3 - \mu a_2^2| \le \frac{2}{7+\alpha}$ -------- (1.5)
Combining (1.2), (1.3), (1.4) and (1.5), we get the required result.
Corollary 2 :- $S^*(f, f', f'', fof) = S^*(f, f', f'')$ as by substituting $\alpha = 1$, the result becomes

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{19}{36} - \frac{4}{9}\mu; \ \mu \le \frac{5}{8}; \\ \frac{1}{4}; \frac{5}{8} \le \mu \le \frac{7}{4}; \\ \frac{4}{9}\mu - \frac{19}{36}; \ \mu \ge \frac{7}{4}. \end{cases}$$

which is the required result given by Gurmeet Singh [21] .

References

[1]. Alexander, J.W *Function which map the interior of unitcircle upon simple regions*, Ann. Of Math., **17** (1995),12-22.
[2]. Bieberbach,L.UberdieKoeffizientemderjenigenPotenzreihen, welcheeineschlichteAbbildung des Einheitskrsisesvermitteln ,*S. – B . Preuss. Akad. Wiss.* **38** (1916), 940-955.
[3]. De Branges L., A proof of Bieberbach Conjecture, *Acta. Math*., **154** (1985),137-152.
[4]. Duren, P.L., Coefficient of univalent functions, Bull. *Amer. Math. Soc.*, **83** (1977), 891-911.
[5]. Fekete, M. and Szegö, G, EineBemerkunguberungeradeschlichtefunktionen, *J.London Math. Soc*., 8 (1933), 85-89.
[6]. Garabedian, P.R. And Schiffer, M., AProof for the Bieberbach Conjecture forthe fourth coefficient, *Arch. Rational Mech. Anal*., 4 (1955), 427-465.
[7]. Kaur, C. and Singh, G., Approach To Coefficient Inequality For A New Subclass Of Starlike Functions With Extremals , *International Journal Of Research In Advent Technology*, **5**(2017) ,
[8]. Kaur, C. and Singh, G., Coefficient Problem For A New Subclass Of Analytic Functions Using Subordination , *International Journal Of Research In Advent Technology*, **5**(2017) ,

[9]. Keogh, F.R. and Merkes , E.P., A coefficient inequality for certain classes of analytic functions , *Proc. Of Amer. Math. Soc.*, **20** (1989), 8-12.

[10]. Koebe, P., Uber Die uniformisiesrungbeliebigeranalyischeerKurven, *Nach. Ges. Wiss.Gottingen* (1907), 633-669.

[11]. Lindelof ,E., Memoire surcertainesinegalitiesdans la theorie des functions monogenes et surquelques proprieties nouvellles de cesfontionsdans la voisinage d′un point singulier essential, *Acta Soc. Sci. Fenn*., **23** (1909) , 481-519.

[12]. Ma ,W. and Minda , D. unified treatment of some special classes of univalent functions , *In Proceedings of the Conference on Complex Analysis* ,Z. Li, F. Ren , I. Yang and S.Zhang (Eds),Int. Press Tianjin (1994) , 157-169.

[13]. Miller, S.S., Mocanu, P.T. And Reade, M.O., All convex functions are univalent and starlike, *Proc. of Amer. Math. Soc.*, 37 (1973), 553-554.

[14]. Nehari, Z. (1952), *Conformal Mappings*, McGraw- Hill, New York.

[15]. Nevanlinna, R., Uber die Eigenshafteneineranalytischenfunkion in der umgebungeinersingularen stele order Linte, *Acta Soc. Sci. Fenn*.,**50** (1922) , 1-46.

[16]. Pederson, R., A proof for the Bieberbach conjecture for the sixth coefficient,*Arch. Rational Mech. Anal*., 31 (1968-69), 331-351.

[17]. Pederson, R. and Schiffer, M., A proof for the Bieberbach conjecture for the fifth coefficient, *Arch. Rational Mech. Anal.*, 45 (1972), 161-193.

[18]. Rani, M., Singh, G., Some Classes Of Schwarzian Functions And Its Coefficient Inequality That Is Sharp, *Turk. Jour. Of Computer and Mathematics Education*, **11** (2020), 1366-1372.

[19]. Rathore, G. S., Singh , G. and Kumawat, L. et.al., Some Subclaases Of A New Class Of Analytic Functions under Fekete-Szego Inequality, *International Journal Of Research In Advent Technology,* **7**(2019) *,*

[20]. Rathore, G. S., Singh , G., Fekete – Szego Inequality for certain subclasses of analytic functions , *Journal Of Chemical , Biological And Physical Sciences,* **5**(2015) *,*

[21]. Singh , G, Fekete – Szego Inequality for a new class and its certain subclasses of analytic functions , *General Mathematical Notes*, **21** (2014),

[22]. Singh , G, Fekete – Szego Inequality for a new class of analytic functions and its subclass, *MathematicalSciences International Research Journal*, 3 (2014),

[23]. Singh, G., Construction of Coefficient Inequality For a new Subclass of Class of Starlike Analytic Functions, *Russian Journal of Mathematical Research Series*, **1** (2015), 9-13.

[24]. Singh, G., Introduction of a new class of analytic functions with its Fekete – Szegö Inequality, *International Journal of Mathematical Archive,* **5** (2014), 30-35.

[25]. Singh, G, An Inequality Of Second and Third Coefficients For A Subclass Of Starlike Functions Constrcted Using Nth Derivative, *Kaav InternationalJournal Of Science, Engineering And Technology*, **4** (2017), 206-210.

[26]. Singh , G, Fekete – Szego Inequality for asymptotic subclasses of family of analytic functions, *Stochastic Modelling And Applications*, 26 (2022),

[27]. Singh , G, Coefficient Inequality For Close To Starlike Functions Constrcted Using Inverse Starlike Classes , *Kaav InternationalJournal Of Science, Engineering And Technology* , **4** (2017) , 177-182.

[28]. Singh , G, Coefficient Inequality For A Subclass Of Starlike Functions That Is Constrcted Using Nth Derivative Of The Functions In The Class , *Kaav InternationalJournal Of Science, Engineering And Technology* , **4** (2017) , 199-202.

[29]. Singh , G, Singh, Gagan, Fekete – Szegö Inequality For Subclasses Of A New Class Of Analytic Functions , *Proceedings Of The World Congress On Engineering* , (2014) , .

[30]. Singh , G, Sarao , M. S. , and Mehrok , B. S., Fekete – Szegö Inequality For A New Class Of Analytic Functions , *Conference Of Information And Mathematical Sciences* , (2013) , .

[31]. Singh , G, Singh, Gagan, Sarao , M. S. , Fekete – Szegö Inequality For A New Class Of Convex Starlike Analytic Functions , *Conference Of Information And Mathematical Sciences* , (2013) , .

[32]. Singh , G, Singh, P.,Fekete – Szegö Inequality For Functions Belonging To A Certain Class Of Analytic Functions Introduced Using Linear Combination Of Variational Powers Of Starlike And Convex Functions, *Journal Of Positive School Psychology* , **6** (2022) , 8387-8391.

[33]. Singh, G. , Fekete – Szegö Inequality For Functions Approaching to A Class In The Limit Form and another Class directly, *Journal Of Information And Computational Sciences*, .

[34]. Singh, G. and Kaur, G., Coefficient Inequality for a Subclass of Starlike Function generated by symmetric points, *Ganita*, **70** (2020), 17-24.

[35]. Singh ,G. and Kaur, G., Coefficient Inequality For A New Subclass Of Starlike Functions, *International Journal Of Research In Advent Technology,* **5**(2017) *,*

[36]. Singh ,G. and Kaur, G., Fekete-Szegö Inequality For A New Subclass Of Starlike Functions, *International Journal Of Research In Advent Technology,* **5**(2017) *,*

[37]. Singh ,G. and Kaur, G., Fekete-Szegö Inequality For Subclass Of Analytic Function Based On Generalized Derivative, *Aryabhatta Journal Of Mathematics And Informatics,* **9**(2017) *,*

[38]. Singh ,G. and Kaur, G., Coefficient Inequality For A Subclass Of Analytic Function using Subordination Method With Extremal Function, *International Journal Of Advance Research In Science And Engineering* , **7** (2018) , .

[39]. Singh ,G. and Garg, J., Coefficient Inequality For A New Subclass Of Analytic Functions, *Mathematical SciencesInternational Research Journal,* **4**(2015) *,*

[40]. Singh ,G. and Kaur, N., Fekete-Szegö Inequality For Certain Subclasses Of Analytic Functions, *Mathematical SciencesInternational Research Journal,* **4**(2015) *,*