International Journal of Research in Engineering and Science (IJRES)
ISSN (Online): 2320-9364, ISSN (Print): 2320-9356
www.ijres.org Volume 10 Issue 10 ǁ October 2022 ǁ PP. 509-513

Study of a Subclass of Classes of P-Valent Starlike and Convex Functions

Divyanshu Verma
Student, GSSDGS Khalsa College, Patiala

ABSTRACT: *We will describe a subclass of p-valent analytic functions in this paper and will obtain sharp upper bounds of the functional* $|a_{p+2} - \mu a_{p+1}^2|$ *for the analytic function* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, |z| < 1$ *belonging to this subclass.*
KEYWORDS: *Univalent functions, Starlike functions, Close to convex functions and bounded functions.*
MATHEMATICS SUBJECT CLASSIFICATION: 30C50

Date of Submission: 08-10-2022 Date of acceptance: 18-10-2022

I. Introduction :

Let $\mathcal{A}$ denote the class of functions of the form
$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ (1.1)
which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([7], [8]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$, Fekete and Szegö [9] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, if\ \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \leq \mu \leq 1; \\ 4\mu - 3, if\ \mu \geq 1. \end{cases} \quad (1.2)$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (See Chhichra [1], Babalola [6]).
Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$$

and satisfying the condition

$$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \quad (1.3)$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A}$$

and satisfying the condition

$$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \quad (1.4)$$

p-VALENT FUNCTION:

Multivalent functions and in particular p-valent functions, are a generalization of univalent functions.In the study of univalent functions, one of the fundamental problems is whether there exists a univalent mapping from a given domain E onto a given domain D. A necessary condition for the existence of such a mapping is that E and D have equal degrees of connectivity. If E and D are simply-connected domains whose boundaries contain more than one point, then this condition is also sufficient and the problem reduces to mapping a given domain onto a disc. In this connection, a special role is played in the theory of univalent functions on simply-connected domains by the S, class of functions f that are regular and univalent on the unit disc $E = \{z:|z| < 1\}$, normalized by the conditions $f(0) = 0, f'(0) = 1$, and having the expansion

$$f(z) = z + a_2z^2 + a_3z^3 + - - -, z \in E$$

In the case of multiply-connected domains, mappings of a given multiply-connected domain onto so-called canonical domains are studied.In particular, p-valent functions can be defined as follow:
Let $\boldsymbol{\mathcal{A}_p}$(p is a positive integer)denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k}\, z^{p+k}$$

which are analytic in the unit disc E. Clearly, $\boldsymbol{\mathcal{A}_1} = \boldsymbol{\mathcal{A}}$. A function $f(z) \in \boldsymbol{\mathcal{A}_p}$is said to be p-valent in E if it assumes no value more than p times in E.

p-VALENT STARLIKE FUNCTION:
A function$f(z) \in \boldsymbol{\mathcal{A}_p}$is said to be a p-valent starlike function in E if there exists a positive real number ρ such that

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0$$

and

$$\int_0^{\pi} \left[Re\left(\frac{zf'(z)}{f(z)}\right)\right] d\theta = 2p\pi, z = re^{i\theta} for$$

$$\rho < |z| < 1.$$

We denote the class of p-valent starlike functions by S_p^*. By $S_p^*(\beta)$, we denote the class of functions $f(z) \in \boldsymbol{\mathcal{A}_p}$satisfying the condition

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \beta; 0 \le \beta < p, z \in E$$

Note: p-valent starlike functions are also called p-valently starlike functions.
$f(z) \in S_p^*(\beta)$ is called p-valently starlike function of order β.
We introduce a new subclass as

$$\left\{f(z) \in \mathcal{A}_p; \frac{[z\{zf'(z)\}']'}{p\{zf'(z)\}'} \prec \frac{1+z}{1-z}; z \in \mathbb{E}\right\}$$

and we will denote this class as $f(z) \in \mathcal{H}_p^*$.
Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z));\ z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form

$$w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1. (1.5)$$

It is known that

$$|d_1| \le 1, |d_2| \le 1 - |d_1|^2 (1.6)$$

II. PRELIMINARY LEMMAS:

For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz}\right)$ so that

$$\frac{1+w(z)}{1-w(z)} = 1 + 2c_1z + 2(c_2 + c_1^2)z^2 + - - - (2.1)$$

Here

$$|c_1| \le 1, |c_2| \le 1 - |c_1|^2 \qquad (2.2)$$

III. MAIN RESULTS

<u>THEOREM 3.1</u>: Let $f(z) \in \mathcal{H}_p^*$, then

$$\left|a_{p+2-\mu a_{p+1}^2}\right| \le \begin{cases} \dfrac{p^3(\,2p+1)}{(p+2)^2} - \dfrac{4up^6}{(p+1)^4}\ if \mu \le \dfrac{(\,p+1\,)^4}{2p^2(p+2)^2} & (3.1) \\ \dfrac{p^3}{(p+2)^2}\ if\ \dfrac{(\,p+1\,)^4}{2p^2(p+2)^2} \le \mu \le \dfrac{(p+1)^5}{2p^3(p+2)^2} & (3.2) \\ \dfrac{4up^6}{(p+1)^4} - \dfrac{p^3(\,2p+1)}{(p+2)^2}\ if\ \mu \ge \dfrac{(p+1)^5}{2p^3(p+2)^2} & (3.3) \end{cases}$$

The results are sharp.

Proof: By definition of $f(z) \in \mathcal{H}_p^*$, we have

$$\frac{[z\{zf'(z)\}']'}{p\{zf'(z)\}'} = \frac{1+z}{1-z}; w(z) \in \mathcal{U}. \qquad (3.4)$$

Expanding the series (3.4), we get

$p^3 z^{p-1} + a_{p+1}(p+1)^3 z^p + a_{p+2}(p+2)^3 z^{p+1} + --- = (1 + 2c_1 z + 2(c_2 + c_1^2)z^2 + --$
$-)(p^3 z^{p-1} + pa_{p+1}(p+1)^2 z^p + pa_{p+2}(p+2)^2 z^{p+1} + ---)$ (3.5)

Identifying terms in (3.5), we get

$a_{p+1} = \frac{2c_1 p^3}{(p+1)^2}$ (3.6)

$$a_{p+2} = \frac{2{c_1}^2 p^4 + (c_2 + c_1^2)p^3}{(p+2)^2} \qquad (3.7)$$

From (3.6) and (3.7), we obtain

$$a_{p+2-}\mu a_{p+1}^2 = \frac{2{c_1}^2 p^4 + (c_2 + c_1^2)p^3}{(p+2)^2} - 4\mu \frac{{c_1}^2 p^6}{(p+1)^4} \qquad (3.8)$$

Taking absolute value, (3.8) can be rewritten as

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{|c_2|p^3}{(p+2)^2} + \left|\frac{2p^4 + p^3}{(p+2)^2} - \mu\ \frac{4p^6}{(p+1)^4}\right| |c_1|^2 \ (3.9)$$

Using (2.2) in (3.9), we get

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{(1-|c_1|^2)p^3}{(p+2)^2} + \left|\frac{2p^4 + p^3}{(p+2)^2} - \mu\ \frac{4p^6}{(p+1)^4}\right| |c_1|^2 \qquad (3.10)$$

<u>Case I</u>:$\mu \le \frac{[2p+1](p+1)^4}{4(p+2)^2 p^3}$

(3.10) can be rewritten as

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{(1-|c_1|^2)p^3}{(p+2)^2} + \left(\frac{2p^4 + p^3}{(p+2)^2} - \mu\ \frac{4p^6}{(p+1)^4}\right)|c_1|^2$$

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{p^3}{(p+2)^2} + \left(\frac{2p^4}{(p+2)^2} - \mu\ \frac{4p^6}{(p+1)^4}\right)|c_1|^2 \qquad (3.11)$$

<u>Subcase I (a)</u>: $\mu \le \frac{P(p+1)^4}{2(p+2)^2 p^3}$

Using (2.2), (3.11) becomes

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{p^3[2p+1]}{(p+2)^2} - \mu \frac{4p^6}{(p+1)^4} \qquad (3.12)$$

<u>Subcase I (b)</u>: $\mu \ge \frac{P(p+1)^4}{2(p+2)^2 p^3}$.

We obtain from (3.11

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{p^3}{(p+2)^2} \qquad (3.13)$$

<u>Case II</u>::$\mu \ge \frac{[2p+1](p+1)^4}{4(p+2)^2 p^3}$

Preceding as in case I, we get

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{p^3}{(p+2)^2} - \left(\frac{2(p^4 + p^3)}{2(p+2)^2} - \mu\ \frac{4p^6}{(p+1)^4}\right)|c_1|^2 \qquad (3.14)$$

<u>Subcase II (a)</u>: $\mu \ge \frac{(p+1)^5}{2(p+2)^2 p^3}$

(3.14) takes the form

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{p^3}{(p+2)^2} - \left(\frac{2(p^4 + p^3)}{2(p+2)^2} - \mu\ \frac{4p^6}{(p+1)^4}\right)$$

$$|a_{p+2} - \mu a_{p+1}^2| \leq \quad \mu \frac{4p^6}{(p+1)^4} - \frac{p^3[2p+1]}{(p+2)^2} \qquad (3.15)$$

Subcase II (b): $\mu \leq \frac{(p+1)^5}{2(p+2)^2 p^3}$

Preceding as in subcase I (b), we get

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{p^3}{(p+2)^2} \qquad (3.16)$$

Combining (3.12), (3.13), (3.15) and (3.16), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by

$f_1(z) \; = \; (1 + az)^h$ where

$$a \; = \quad \frac{2p^3(p+2)^2 - 2(p+1)^4 p + 1)}{(p+1)^2 (p+2)^2}$$

And

$$h = \frac{2p^3(p+2)^2}{2p^3(p+2)^2 - (p+1)^4(2p+1)}$$

Extremal function for (3.2) is defined by

$$f_2(z) = z\left(1 + \frac{p^3 z}{2(p+2)^2}\right)^2$$

Corollary 3.4: Putting $p = 1$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{3} - \frac{\mu}{4} \; if \;\; \mu \leq \frac{8}{9} \\ \frac{1}{9} \; if \quad \frac{8}{9} \leq \mu \leq \frac{16}{9} \\ -\frac{1}{3} + \frac{\mu}{4} \; if \; \mu \geq \frac{16}{9} \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent convex functions.

References:

[1]. Alexander, J.W Function which map the interior of unitcircle upon simple regions, Ann. Of Math., **17** (1995),12-22.

[2]. Bieberbach, L. Uber die Koeffizientem derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskrsises vermitteln , S. – B . Preuss. Akad. Wiss. **38** (1916), 940-955.

[3]. De Branges L., A proof of Bieberbach Conjecture, Acta. Math., **154** (1985),137-152.

[4]. Duren, P.L., Coefficient of univalent functions, Bull. Amer. Math. Soc., **83** (1977), 891-911.

[5]. Fekete, M. and Szegö, G, Eine Bemerkung uber ungerade schlichte funktionen, J.London Math. Soc., 8 (1933), 85-89.

[6]. Garabedian, P.R., Schiffer, M., AProof for the Bieberbach Conjecture forthe fourth coefficient, Arch. Rational Mech. Anal., 4 (1955), 427-465.

[7]. Kaur, C. and Singh, G., Approach To Coefficient Inequality For A New Subclass Of Starlike Functions With Extremals, International Journal Of Research In Advent Technology, **5**(2017)

[8]. Kaur, C. and Singh, G., Coefficient Problem For A New Subclass Of Analytic Functions Using Subordination, International Journal Of Research In Advent Technology, **5**(2017)

[9]. Kaur. G, Singh. G, Arif. M, Chinram. R, Iqbal. J, A study of third and fourth Hankel determinant problem for a particular class of bounded turning functions, Mathematical Problems in Engineering,22, 511-526, 2021

[10]. Keogh, F.R., Merkes, E.P.,A coefficient inequality for certain classes of analytic functions, Proc. Of Amer. Math. Soc., **20**, 8-12, 1989.

[11]. Koebe, P., Uber Die uniformisiesrung beliebiger analyischeer Kurven, Nach. Ges. Wiss.Gottingen (1907), 633-669.

[12]. Lindelof, E., Memoire sur certaines inegalities dans la theorie des functions monogenes et sur quelques proprieties nouvellles de ces fontions dans la voisinage d′un point singulier essential, Acta Soc. Sci. Fenn., **23** (1909) , 481-519.

[13]. Ma, W. and Minda , D. unified treatment of some special classes of univalent functions, In Proceedings of the Conference on Complex Analysis , Int. Press Tianjin (1994) , 157-169.

[14]. Mehrok. B. S, Singh. G,Saroa. M. S,Fekete-Szegö Inequality for Certain Subclasses of Analytic Functions, Acta Ciencia Indica, 39 (2), 97-104, 2013

[15]. Mehrok. B. S, Singh. G,Saroa. M. S,Fekete-Szegö Inequality for a Certain Sub-classes of Analytic Functions, Acta Ciencia Indica, 39 (2), 125-138, 2013

[16]. Mehrok. B. S, Singh. G,Saroa. M. S,Fekete-Szegö Inequality for a Certain Sub-classes of Analytic Functions, Acta Ciencia Indica, 39 (3), 217-228, 2013

[17]. Miller, S.S., Mocanu, P.T. And Reade, M.O., All convex functions are univalent and starlike, Proc. of Amer. Math. Soc., 37 (1973), 553-554.

[18]. Nehari, Z. (1952), Conformal Mappings, Mc Graw- Hill, New York.

[19]. Nevanlinna, R., Uber die Eigenshaften einer analytischen funkion in der umgebung einer singularen stele order Linte, Acta Soc. Sci. Fenn.,50 (1922), 1-46.

[20]. Pederson, R., A proof for the Bieberbach conjecture for the sixth coefficient,Arch. Rational Mech. Anal., 31 (1968-69), 331-351.

[21]. Pederson, R. and Schiffer, M., A proof for the Bieberbach conjecture for the fifth coefficient, Arch. Rational Mech. Anal., 45 (1972), 161-193.

[22]. Rani, M., Singh, G., some classes of Schwarzian functions and its coefficient inequality that is sharp, Turk. Jour. Of Computer and Mathematics Education, **11** (2020), 1366-1372.

[23]. Rathore, G. S., Singh, G. and Kumawat, L. et.al., Some Subclaases Of A New Class Of Analytic Functions under Fekete-Szego Inequality, Int. J. of Res. In Adv. Technology, **7**(2019)

[24]. Rathore. G. S., Singh, G., Fekete – Szego Inequality for certain subclasses of analytic functions , Journal Of Chemical , Biological And Physical Sciences, **5**(2015) ,

[25]. Singh. G, Fekete – Szego Inequality for a new class and its certain subclasses of analytic functions , General Mathematical Notes, **21** (2014),

[26]. Singh. G, Fekete – Szego Inequality for a new class of analytic functions and its subclass, MathematicalSciences International Research Journal, 3 (2014),

[27]. Singh. G., Construction of Coefficient Inequality For a new Subclass of Class of Starlike Analytic Functions, Russian Journal of Mathematical Research Series, **1** (2015), 9-13.

[28]. Singh, G., Introduction of a new class of analytic functions with its Fekete – Szegö Inequality, International Journal of Mathematical Archive, **5** (2014), 30-35.

[29]. Singh. G, An inequality of second and third coefficients for a subclass of starlike functions constrcted using nth derivative, Kaav Int.J. of Sci. Eng. And Tech., **4**(2017), 206-210.

[30]. Singh, G, Fekete – Szego Inequality for asymptotic subclasses of family of analytic functions, Stochastic Modelling And Applications, 26 (2022),

[31]. Singh, G, Coefficient Inequality For Close To Starlike Functions Constrcted Using Inverse Starlike Classes , Kaav Int.J. of Sci. Eng. And Tech., **4** (2017), 177-182.

[32]. Singh, G, Coeff. Inequality for a subclass of Starlike functions that is constructed using nth derivative of the fns in the class, Kaav Int.J. Of Sci. Eng. And Tech., **4** (2017), 199-202.

[33]. Singh, G., Fekete – Szegö Inequality for functions approaching to a class in the limit form and another class directly, Journal Of Information And Computational Sciences, 12 (4), 2022, 181-186

[34]. Singh, G., Garg, J., Coefficient Inequality For A New Subclass Of Analytic Functions, Mathematical SciencesInternational Research Journal, **4**(2015)

[35]. Singh, G, Singh, Gagan, Fekete – Szegö Inequality For Subclasses Of A New Class Of Analytic Functions , Proceedings Of The World Congress On Engineering , (2014) , .

[36]. Singh, G, Sarao, M. S., and Mehrok, B. S., Fekete – Szegö Inequality For A New Class Of Analytic Functions , Conference Of Information And Mathematical Sciences , (2013).

[37]. Singh. G, Singh. Gagan, Sarao. M. S., Fekete – Szegö Inequality for A New Class of Convex Starlike Analytic Functions, Conf. Of Information and Mathematical Sciences, (2013).

[38]. Singh, G., Kaur, G., Coefficient Inequality for a Subclass of Starlike Function generated by symmetric points, Ganita, **70** (2020), 17-24.

[39]. Singh ,G., Kaur, G., Coefficient Inequality For A New Subclass Of Starlike Functions, International Journal Of Research In Advent Technology, **5**(2017) ,

[40]. Singh, G., Kaur, G., Fekete-Szegö Inequality For A New Subclass Of Starlike Functions, International Journal Of Research In Advent Technology, **5**(2017) ,

[41]. Singh, G., Kaur, G., Fekete-Szegö Inequality For Subclass Of Analytic Function Based On Generalized Derivative, Aryabhatta Journal Of Mathematics And Informatics, **9**(2017) ,

[42]. Singh, G., Kaur, G., Coefficient Inequality For a subclass of analytic function using subordination method with extremal function, Int. J. Of Advance Res. In Sci & Engg , **7** (2018)

[43]. Singh, G., Kaur, G., Arif, M., Chinram R, Iqbal J, A study of third and fourth Hankel determinant problem for a particular class of bounded turning functions, Mathematical Problems in Engineering, 2021

[44]. Singh, G. and Kaur, G., 4th Hankel determinant for α bounded turning function, Advances in Mathematics: Scientific Journal, 9 (12), 10563-10567

[45]. Singh, G., Kaur, N., Fekete-Szegö Inequality For Certain Subclasses Of Analytic Functions, Mathematical SciencesInternational Research Journal, **4**(2015)

[46]. Singh, G, Singh, B, Fekete Szego Coefficient Inequality of Regular Functions for A Special Class, International Journal of Research in Engineering and Science, 10 (8), 2022, 556-560

[47]. Singh, G, Singh, P., Fekete – Szegö Inequality For Functions Belonging To A Certain Class Of Analytic Functions Introduced Using Linear Combination Of Variational Powers Of Starlike And Convex Functions, Journal Of Positive School Psychology, **6** (2022), 8387-8391.

[48]. Singh. G, Rani M, An advance subclass of Analytic Functions having a unique coefficient inequality, Int. J. of Research in Engineering and Science, 10 (8), 2022, 474-476

[49]. Singh, G., Singh, G., Singh, G., A subclass of bi-univalent functions defined by generalized Sãlãgean operator related to shell-like curves connected with Fibonacci numbers, International Journal of Mathematics and Mathematical Sciences, 2019

[50]. Singh, G., Singh, G., Singh, G., Certain subclasses of univalent and biunivalent functions related to shell-like curves connected with Fibonacci numbers, General Mathematics, 28 (1), 125-140, 2020

[51]. Singh, G., Singh, G., Singh, G., Certain subclasses of Sakaguchitype bi-univalent functions, Ganita, 69 (2), 45-55, 2019

[52]. Singh, G., Singh, G., Singh, G., Certain Subclasses of Bi-Close-to-Convex Functions Associated with Quasi-Subordination, Abstract and Applied Analysis, 1, 1-6, 2019

[53]. Singh, G., Singh, G., Singh, G., Fourth Hankel determinant for a subclass of analytic functions defined by generalized S˘al˘agean operator, Creat. Math. Inform., 31(2), 229-240, 2022

[54]. Srivastava H. M., G. Kaur, Singh. G, Estimates of fourth Hankel determinant for a class of analytic functions with bounded turnings involving cardioid domains, Journal of Nonlinear and Convex Analysis, 22 (3), 511-526, 202