Numerical Solution for the Design of a Ducted Axisymmetric Nozzle using Method of Characteristics

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Abstract: Supersonic nozzles find application in the field of Rocket Propulsion. A method for the design of a ducted axisymmetric nozzle for high speed, low density flows is described and this work was carried out in the rules of Aerodynamics with the aim of projecting a computational method for the calculation of a Ducted Axisymmetric nozzle of minimum length, using the method of characteristics, by expanding a flow of air until to the required Mach Number. A thorough understanding of the method of characteristics and its application to the design of a Ducted Axisymmetric nozzle is required. We make use of the compatibility equations involved in the axisymmetric method of characteristics, Prandtl-Meyer function and develop a MATLAB program with the help of which the contour of the ducted axisymmetric nozzle has been developed for an exit Mach number 2.5, with an output of 4 expansions. A similar code was also developed, but for the two-dimensional case for 30 expansions as it more was simpler. Finally, the desired exit Mach number has been achieved giving a minimum length of the nozzle with a shock free and isentropic flow. We will be dealing with the various aspects of the draft code and its implementation, as well as the results obtained.

Keywords: Method of characteristics, Characteristic lines, Ducted Axisymmetric, Minimum Length Nozzle, Prandtl-Meyer Function, Finite Differences Method.

Nomenclature

x	-	x co-ordinate of throat.
r	-	r co-ordinate of throat.
A	-	Throat area.
v	-	Prandtl-Meyer function.
M_e	-	Exit Mach number or design Mach number.
μ	-	Mach Angle.
θ	-	Deflection Angle.
θ_{max}	-	Expansion angle of the wall downstream of the throat.
V	-	Velocity of the flow.

I. INTRODUCTION

Axisymmetric supersonic nozzles are inevitable in rocket nozzles, propulsive systems and mixing devices. Conventional applications of supersonic nozzles include experimental facilities like supersonic wind tunnels, missiles, variable area nozzles, gas dynamic lasers, molecular beams, air guns, re-entry vehicles etc. Challenging endeavors such as reusable launch vehicles require nozzles to operate in various flow regimes without compromise on their performance like thrust loss. Hypersonic air breathing vehicles are being vigorously pursued by several countries and space agencies. These applications would benefit from active modification of nozzle contours, since the aerodynamic range of operation is vast, and since the computational capabilities and possibilities are enormous. In all these applications, the shape of the nozzle contour is extremely

Some of the aforementioned applications may demand active modifications to the nozzle contour, based on closed-loop feedback. Such active modifications would involve real-time computation of the nozzle contour, and would hence demand efficient nozzle design methods. Therefore, this paper focuses on design methods for supersonic nozzle contours.This work was carried out with the aim of implementing a computational method that draw the contour of a Ducted Axisymmetric minimum length nozzle to expand the air flow up to a Mach number of 2.5, based on the method of characteristics. However, because the initial difficulties in implementation of the calculation for a Ducted Axisymmetric, began to carry out this work for a Ducted 2D.

II. DEVELOPMENT OF COMPUTATIONAL METHOD

2.1. GENERAL ANALYSIS

Once that, for the case of Axisymmetric, the equations of the lines characteristics are more complex than in two-dimensional and they have to resort to the numerical methods for their resolution. Once we have found the method for resolution of numerical equations, it is necessary to apply it to the entire fabric taking into account the conditions frontiers, both on the axis of symmetry of the nozzle, either on the wall. Thus, the structures of the code developed, as well as all the functions are more relevant.

2.2 ANALYSIS OF THE FUNCTION INTERSECT.M.

The intersect function m is responsible for the calculation of points within the mesh. Let us consider 2 points, one from a C_- and another from a C_+ and the function, according to the following equations presented, must determine its intersection within the lattice, finding the new point.

As specified in reference [3], for the application of the method of characteristics in Axisymmetric, Irrotational flow uses the following relations:

$$d(\theta + v) = \frac{1}{\sqrt{M^2 - 1} - \cot\theta} \frac{dr}{r}$$

$$(Along\ C_+ Characterisitcs\ )$$

$$\left(\frac{dr}{dx}\right)_{c_+} = \tan(\theta + \mu) \qquad ---- (1)$$

$$d(\theta - v) = \frac{-1}{\sqrt{M^2 - 1} + \cot\theta} \frac{dr}{r}$$

$$(Along\ C_- Chatacteristic\ )$$

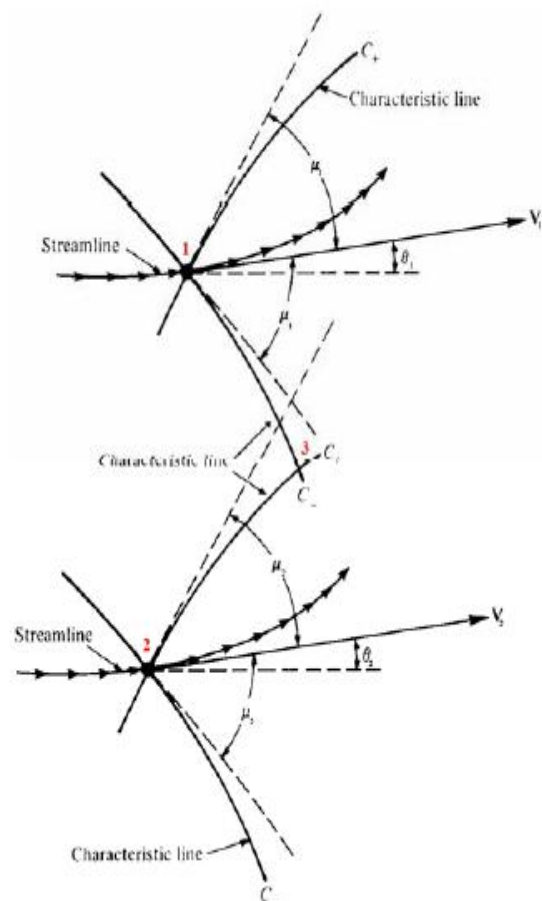$$\left(\frac{dr}{dx}\right)_{c_-} = \tan(\theta - \mu) \qquad ---- (2)$$

Figure.1: *Features of the intersection of Characteristic Lines*

The numerical integration of the equations can be made by using the method of finite differences to obtain the following discretized equations:

$$(\theta_3 - \theta_2) + (v_3 - v_2 = \frac{1}{\sqrt{M_3{}^2 - 1} - \cot\theta_3} \frac{r_3 - r_2}{r_3} \quad (Along\ C_+ Characteristic) \qquad ---- (3)$$

$$(\theta_3 - \theta_1) - (v_3 - v_1) = \frac{-1}{\sqrt{M_3{}^2 - 1} + \cot\theta_3} \frac{r_3 - r_2}{r_3} \quad (Along\ C_- Characteristic) \qquad ---- (4)$$

and

$$\frac{r_3 - r_1}{x_3 - x_1} = \tan(\theta_1 - \mu_1) \qquad ---- (5)$$

$$\frac{r_3 - r_2}{x_3 - x_2} = \tan(\theta_2 + \mu_2) \qquad \text{---- (6)}$$

These discretized equations are arranged in order to obtain the values of θ_3 and v_3, once the properties at points 1 and 2 are known

$$\theta_3 = \frac{\theta_1 + \theta_2 + v_1 - v_2 + \dfrac{1}{\sqrt{M_3{}^2 - 1} - \cot\theta_3}\dfrac{r_3 - r_1}{r_3} - \dfrac{1}{\sqrt{M_3{}^2 - 1} + \cot\theta_3}\dfrac{r_3 - r_2}{r_3}}{2} \qquad \text{---- (7)}$$

$$v_3 = \frac{\theta_1 - \theta_2 + v_1 + v_2 + \dfrac{1}{\sqrt{M_3{}^2 - 1} - \cot\theta_3}\dfrac{r_3 - r_1}{r_3} - \dfrac{1}{\sqrt{M_3{}^2 - 1} + \cot\theta_3}\dfrac{r_3 - r_2}{r_3}}{2} \qquad \text{---- (8)}$$

Initially θ_3 was determined through an iterative process, since the equation 7 is an implicit equation. In this way, θ_3 arbitrates is to be used in cot (θ_3), r_3 and M_3, which is the average between θ_1 and θ_2, between r_1 and r_2, and between M_1 and M_2 respectively, until a value of θ_3 obtained. Once the values θ_3 and v_3 are obtained based on equations 7 and 8 and by Prandtl-Meyer equation, knowing the value of M_3, it will be used in the recalculation of θ_3. However, we must still calculate the new value of r_3 based on the x_3. So an iterative method was implemented which will end when the difference between the previous and the following when r3 is of the order 1×10^{-5}. In short, the intersect function. m makes use of an iterative method for calculating θ_3 for the exterior and r_3.

In the end, the intersect function.m returns the properties of the point, this is: x_3, r_3, θ_3, v_3, M_3 and μ_3.

$$x_3 = \frac{r_2 - r_1 + \tan(\theta_1 - \mu_1)x_1 - \tan(\theta_2 - \mu_2)x_2}{\tan(\theta_1 - \mu_1) - \tan(\theta_2 - \mu_2)} \qquad \text{---- (9)}$$

$$r_3 = r_1 + (x_3 - x_1)\tan(\theta_1 - \mu_1) \qquad \text{---- (10)}$$

2.3 ANALYSIS OF THE FUNCTION WALL.M
In relation to the points on the wall of nozzle, and as in two-dimensional case, there's no reflection of expansion wave on the wall and in order to obtain the conditions of minimum length nozzle, one has to impose the values of variables which do not change from the previous point to the point on the wall, in particular, θ, v, μ and M.
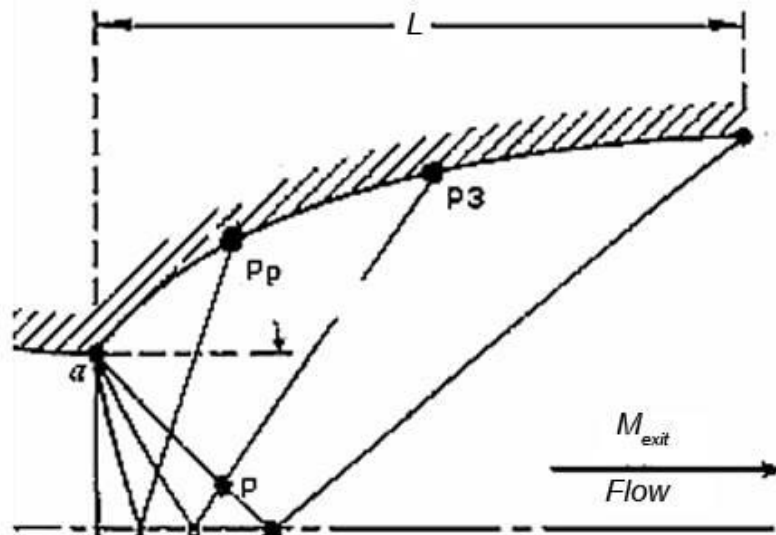

Figure.2: Incidence of the Characteristic lines on the wall features.

In this format, we have to calculate the coordinates of the new point, P_3, which refers to the intersection of the straight line containing the point P_p, where the slope is tan $((\theta_{Pp} + \theta_{P3})/2)$, with the straight line leaves the point P whose slope is given by tan $(\theta + \mu)$, which is the slope point feature.

So, given the general equation of a straight line: $r - r_o = m(x - x_o)$ and replacing it if

$$r - r_{p_p} = \tan\left(\frac{\theta_{p_p} + \theta_3}{2}\right) . (x - x_{p_p}) \qquad \text{---- (11)}$$

$$r - r_p = \tan(\theta_p + \mu_p) . (x - x_p) \qquad \text{---- (12)}$$

Basically the function Wall.m receives all the data points P and P_p, respectively, and, on the basis described in the previous paragraph, returns all the data in the point on the wall, P_3.

2.4 ANALYSIS OF THE FUNCTION AXIS.M.

The function Axis.m calculates the points of intersection of the lines which feature with the axis of symmetry. For the calculation of points on the axis of the nozzle resorted to equation (4), through the characteristic C_ that descends to intersect the said axis. Knowing the values of θ, υ and r to the point from which the characteristic line and the value of θ and r are on the axis of symmetry, we obtain an equation just dependent on υ and M. υ being a function of M, it is possible, through an iterative process, to obtain the values of these two variables.

$$v_3 = \theta_1 - \theta_3 + v_1 + \frac{1}{\sqrt{M_3^2 - 1} - \cot\theta_3} \frac{r_3 - r_1}{r_3} \qquad \text{---- (13)}$$
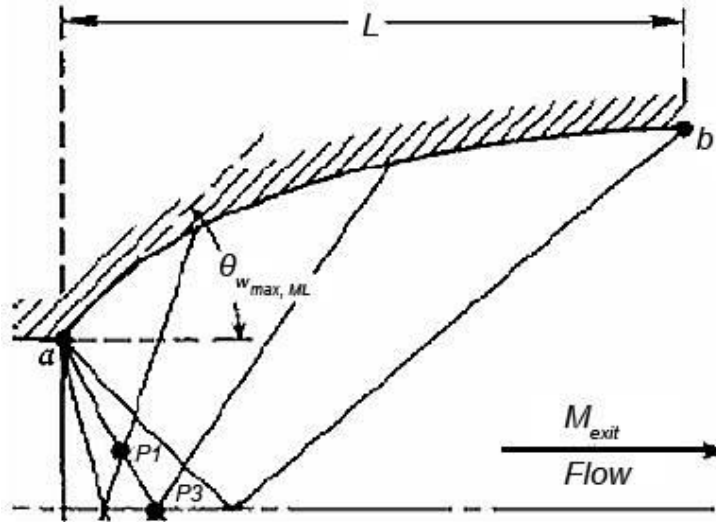

Figure.3: *Intersection of the Characteristic lines on the Axis of symmetry.*

2.5. ANALYSIS OF THE FUNCTION MAIN.M

The program main.m consists of the main program, which makes use of functions previously presented, in order to effectively draw the nozzle contour, calculating the coordinates of the points of the wall and the intersections of the features.The input parameters required are:
• Gamma;
• Exit Mach number of nozzle;
• Number of increments in the range of expansion;
•Action RADIUS of nozzle throat;
• Initial increment angle, the first expansion;

Beyond these parameters there is another parameter which is the distance to the axis of the nozzle. Ideally this value should be 0. However, if that were the case, the calculation of points on the axis, when applying the characteristic line equation (13), would differ to infinity, since the RADIUS appears in the denominator.

Defining the parameters, which will calculate the slope of the wall at point, a, corresponding to θ_{max} is:

$$\theta_{max} = \frac{v(Mach\_Exit)}{2} \qquad \text{---- (14)}$$

With this value it is possible to calculate the value of the angle between the various increments of expansion, after the initial increment:

$$\Delta\theta = \frac{\theta_{max} - InitialIncrements}{No.ofIncrements - 1} \qquad \text{---- (15)}$$

Before we start calculating the points, calculate the total no. of existing points depending on the number of increments of the expansion.

Then, it begins to build the matrices main_a and main_p. The first is referring only to point a, corresponding to each line at each increment of the expansion, while the second relates to all the other points. Both arrays have 6 columns that represent x,r,θ, ν, M and μ, respectively.

Thus, in the matrix main_a array, all values of x are 0 and all values ofr correspond to the throat RADIUS. The values of θ are obtained from the angle ofinitial increment and incrementing it Δθ along the expansion and, because ofinitial expansion, υ = θ. Finally, the Mach is calculated with the function invpran.m, using thevalue of υ previously determined, and μ is obtained by the equation:

$$\mu = \sin^{-1}\left(\frac{1}{M}\right) \qquad \text{---- (16)}$$

In relation to the main_p array, this is calculated in two stages. First the point 1 and after all others.

For point 1, the values of r are the values of the radius of the axis, θ the initial increment, as well as ν, and the values of Mach and μ are calculated in the same way that has been described for the array main_a. In relation to x, this is obtained as follows:

$$x = r - \frac{r_a}{\tan\left(\frac{1}{2}(\theta_a + \theta) - \frac{1}{2}(\mu_a + \mu)\right)} \qquad \text{---- (17)}$$

The rest of the points in the array main_p are obtained through a cycle. This cycle consists of the following: since the point 1 has already been calculated, it begins the calculation of point 2, the first expansion. Through a cycle, run through the various points of the current expansion, using the function intersect.m for its calculation, up to the point before the wall.

When we get to the wall using the function wall.m, being that in the case of the first expansion is necessary to provide as input data the point a and the previous point, while that, in the case of another expansion, provides the previous wall point and the previous point feature.

Having calculated the point on the wall, check if the current increment corresponds to last expansion. If yes, the calculation is completed. Otherwise, the expansion follows.Next, we calculate the intersection in the central axis of this expansion and back to the beginning of the cycle.

III.   ANALYSIS OF RESULTS.

The output of expansions for a Mach number of 2.5 and 4, we obtain the following results:
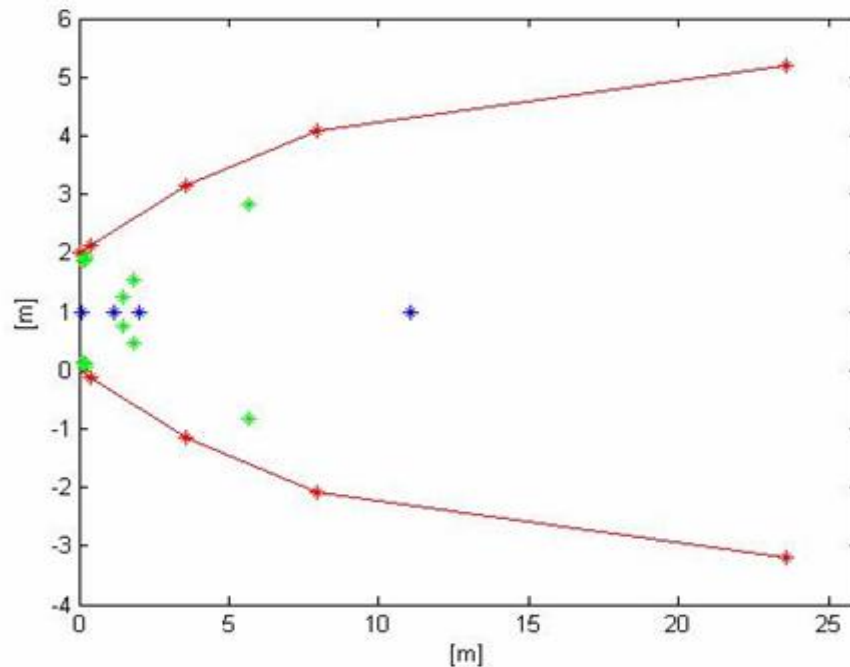
Figure. 1.*Contour design of the nozzle for 4 expansions. The red dots correspond to the points of wall, the Blues to the points of the axis and the Greens to interior points.*

The maximum number of features for which the program has been running, was four because from then on convergence errors were obtained, by that it is not possible to draw the entire nozzle.

Although if we want to obtain a Mach 2.5, the output was a little higher than expected about 20% higher due to numerical errors due to the use of finite differences. However, if you compare the contour that was obtained with the outline of the nozzle reference [1] turns out that is very similar.
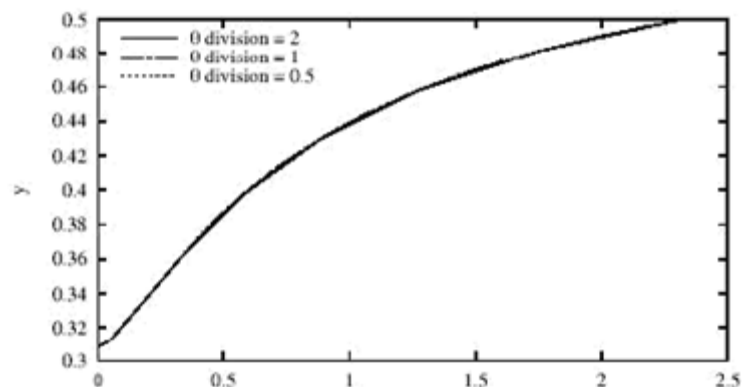
Figure. 2:*Outline of Axisymmetric nozzle for Mach 2.5, according to reference [1]*

In order to develop the contour as shown in the Figure1, we develop a separate code in which the co-ordinates of the nozzle obtained from the previous codes, are used along with the spline tool box to obtain the contour of the nozzle as shown in the Figure 3. The curve has been generated within the ratio 1:8.8 across the coordinate axis and then the set of 4 points of the nozzle are joined with the help of a 3D-splinecurve, creating a smooth profile.

The curve is constructed is such a way that, first we generate the points of the nozzle and then by appending the first point to the end of the last point using the spline tool box, we get a smooth curve as shown below.
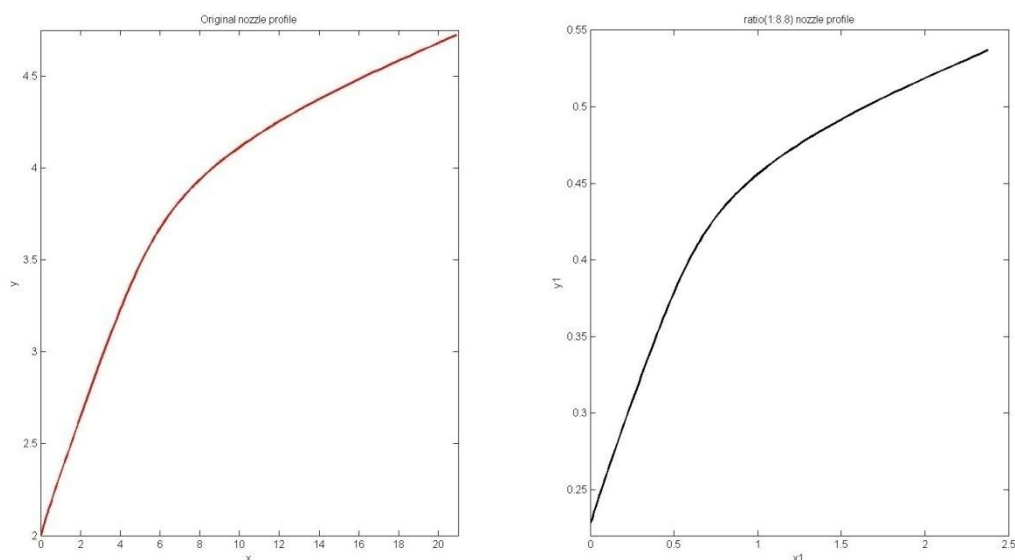

Figure. 3: *Contour of Axisymmetric nozzle for Mach 2.5 generated with the help of Spline toolbox*

3.1. COMMENTARY ON CODE FOR 2D-NOZZLE.

As mentioned above a similar code was also developed, but for the two-dimensional case, it more was simpler. So, based on this code, yielding the results represented in Figure 3 for 30 expansions. This proved to be more stable and robust than the program developed for the calculation of nozzle in axisymmetric, allowing further variations in input parameters.
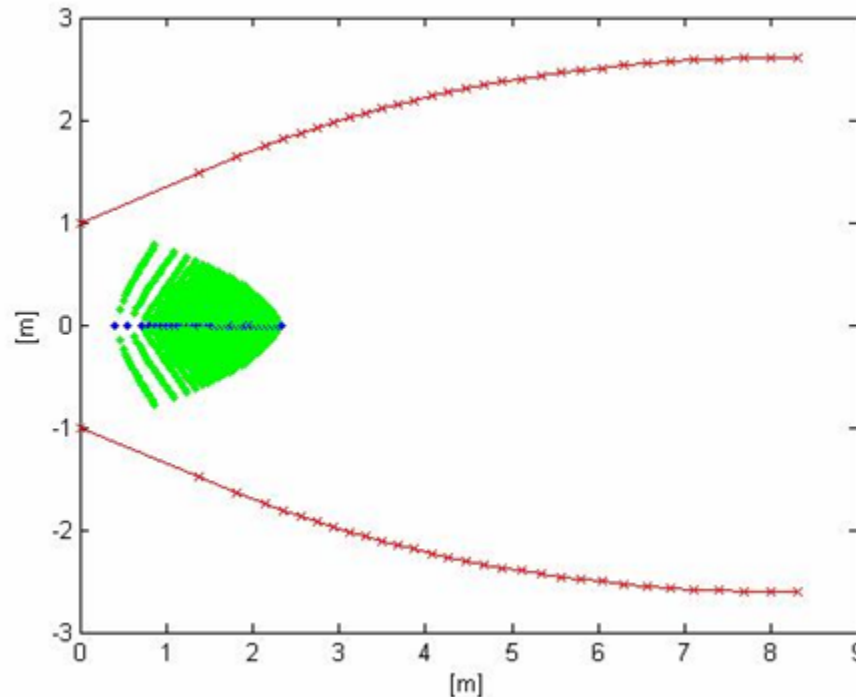
Figure. 3. *The contour of the nozzle for 30expansions. The red dots correspond to points of the wall, blue dots to axis points and the green dots to the interior points.*

IV.   CONCLUSION

Supersonic convergent-divergent nozzles (SCDN) have many applications. They are usually subjected to complex flow pattern. Different degrees of simplification have been studied. Computer is a must to achieve high accuracy and large calculation needed with modern high speed applications. Hence, a computerized approximation approach might be a better method to tackle such a problem. Characteristic method is the most appropriate method to be used with (SCDN) design. Programming of this method directly is impractical as it uses a step-by-step graphical procedure.

After the completion of this study, it is concluded that the computational method implemented for the ax symmetric case is not as robust as one would expect. Nevertheless we can obtain quite reasonable results to the academic level of the work. On the other hand, it is concluded that the programming tool used, MATLAB, proved to be quite accessible given the simplicity with which program. In short, the completion of this work, despite its difficulty quite high, permitted to acquire knowledge and skills in the area of finite differences and the use of computational methods for the study of physical phenomena.

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