International Journal of Research in Engineering and Science (IJRES)
ISSN (Online): 2320-9364, ISSN (Print): 2320-9356
www.ijres.org Volume 1 Issue 8 ǁ Dec 2013 ǁ PP.42-47

Solution of Fractional Order Stokes´ First Equation

Ahmad El-Kahlout
Al-Quds Open University, North Gaza Branch, Beit Lahia, Palestine.

Abstract: Fractional sine transform and Laplace transform are used for solving the Stokes` first problem with rational derivative, where the fractional derivative is defined in the Caputo sense of order $m-1<\alpha\le m$.
The solution of classical problem for Stokes` first problem has been obtained as limiting case.
Keywords: : Stokes` first problem, Fractional derivatives, Laplace transform, Fourier sine transform, Caputo fractional derivative.

I. Introduction

Fractional partial differential equations have many applications in applied sciences and engineering. These applications appear in gravitation elastic membrane, electrostatics, fluid flow, steady state, heat conduction and many other topics in both pure and applied mathematics. Typical examples of fractional of fractional partial differential equations of the time fractional advection dispersion equation as in[6,7], fractional diffusion equation as in[16,8,5,9,15], fractional wave equation as in[14]. The Rayleigh-stokes fractional equations as in[2].

The Stokes fractional equations are examples of fractional partial differential equation .

In this paper we consider Stokes´ first fractional equation for the flat plate. Exact solution of this equation will be investigated. The Fourier sine transform and fractional Laplace transform are used for getting exact solution for this equation. The fractional terms in Stokes´ equation are considered as Caputo fractional derivative.
Basic Definitions:

Definition1: The Rieman-Liouville fractional integral[10,2] of order α is defined as:

e

Definition 2: The Caputo fractional derivative [10] of order $m-1<\alpha\le m$ is defined as:

$$D^{\alpha} f(x)=\frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-m+1}}\,dt$$

Definition 3: The Laplace integral transform[11,13,4,10], of the function $f(x)$ is defined as:

$$L(f(x))=\int_0^{\infty} f(x)\,e^{-st}dx$$

Definition 4: The Fourier sine integral transform[4,10,1], of the function $f(x)$ is defined as:

$$F_e(f(x))=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\sin(\zeta x)\,f(x)dx$$

II. Solution of Stokes` first problem

Consider the Stokes´ first problem for a heated flat plate

$$\frac{\partial u(x,t)}{\partial t}=\left(\nu+\alpha D_t^{\beta}\right)\frac{\partial^2 u(x,t)}{\partial x^2} \qquad (1)$$

where $u(x,t)$ is the velocity, *t* is the time, *x* is the distance and ν, α are constants with respect to *x* and *t* and D_t^{β} is the Caputo fractional derivative with $m-1<\beta\le m$. The corresponding initial and boundary conditions of Eq.(1) are

$$\frac{\partial^n u(x,0)}{\partial t^n}=b_n(x),\ n\ge 0\ ,\quad x>0 \qquad (2)$$

$$u(0,t)=U, \quad for \quad t>0 \tag{3}$$

Moreover, the natural condition

$$u(x,t), \frac{\partial^n u(x,t)}{\partial x^n} \to 0, n \ge 0 \quad and \quad x \to \infty \tag{4}$$

also has to be satisfied.
Employing the non-dimensional quantities

$$u^* = \frac{u}{U}, x^* = \frac{xU}{\nu}, t^* = \frac{tU^2}{\nu}, \eta = \alpha \frac{U^2}{\nu^2} \tag{5}$$

Now $u = u^* U$, then

$$\frac{\partial u}{\partial t} = \frac{\partial u^*}{\partial t} U + \frac{\partial U}{\partial t} u^* \tag{6a}$$

but since $u(0,t) = U = 1$, then

$\frac{\partial u}{\partial t} = 0, \quad for\, t \ge 0$, then $\frac{\partial u}{\partial t} = \frac{\partial u^*}{\partial t}$

Now

$$\frac{\partial u^*}{\partial t} = \frac{\partial u^*}{\partial t^*} \frac{\partial t^*}{\partial t} = \frac{\partial u^*}{\partial t^*} \frac{1}{\nu}, \tag{6b}$$

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial u^*}{\partial x} \tag{6c}$$

and

$$\frac{\partial u^*}{\partial x} = \frac{\partial u^*}{\partial x^*} \frac{\partial x^*}{\partial x} = \frac{1}{\nu} \frac{\partial u^*}{\partial x^*} \tag{6d}$$

then

$$\frac{\partial u(x,t)}{\partial x} = \frac{1}{\nu} \frac{\partial u^*}{\partial x^*} \tag{6f}$$

and

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{\nu^2} \frac{\partial^2 u^*}{\partial x^{*2}} \tag{6h}$$

Eqs. (1) to (5) reduce to dimensionless equations as follows (for brevity the dimensionless mark “*” is omitted here)

$$\frac{\partial u(x,t)}{\partial t} = \left(1 + \eta D_t^\beta\right) \frac{\partial^2 u(x,t)}{\partial x^2}, \quad m-1 < \beta \le m, \quad \eta = \frac{\alpha}{\nu} \tag{7}$$

$$\frac{\partial^n u(x,0)}{\partial t^n} = b_n(x), n \ge 0, \; for \quad x > 0 \tag{8}$$

$$u(0,t) = 1, \quad t > 0 \tag{9}$$

$$u(x,t), \frac{\partial^n u(x,t)}{\partial x^n} \to 0, n \ge 0 \quad for \quad x \to \infty \tag{10}$$

Making use the Fourier sine integral transform and boundary conditions (9), (10). Then Eqs. (7) and (8) leads to

$$\frac{\partial U(\zeta,t)}{\partial t} = -\zeta^2 \left(1 + \eta D_t^\beta\right) U(\zeta,t) + \sqrt{\frac{2}{\pi}} \zeta \tag{11}$$

$$U^{(n)}(\zeta,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(\zeta x) b_n(x) dx = b_n(\zeta), n \ge 0 \tag{12}$$

Hence the Laplace transform of Eq. (11)is

$$p\widetilde{U}(\zeta,p)+\zeta^2\widetilde{U}(\zeta,p)+\eta\zeta^2p^\beta\widetilde{U}(\zeta,p)-\sum_{n=0}^{m}\eta\zeta^2p^{\beta-n-1}U^{(n)}(\zeta,0)=\sqrt{\frac{2}{\pi}}\frac{\zeta}{p} \quad (13)$$

$$\widetilde{U}(\zeta,p)=\frac{\sqrt{\frac{2}{\pi}}\zeta}{p\left(p+\zeta^2\eta p^\beta+\zeta^2\right)}+\sum_{n=0}^{m}\frac{\zeta^2\eta b_n(\zeta)p^{\beta-n-1}}{p+\zeta^2\eta p^\beta+\zeta^2} \quad (14)$$

$$\widetilde{U}(\zeta,p)=\frac{\sqrt{\frac{2}{\pi}}\zeta}{p\left(\zeta^2\eta p^\beta+p+\zeta^2\right)}+\sum_{n=0}^{m}\frac{\zeta^2\eta b_n(\zeta)p^{\beta-n=1}}{\zeta^2\eta p^\beta+p+\zeta^2} \quad (15)$$

Taking the inverse Laplace transform of Eq. (15) and using the relation

$$L^{-1}\left\{\frac{n!\,p^{\lambda-\mu}}{\left(p^\lambda\mp c\right)^{n+1}}\right\}=t^{\lambda n+\mu-1}E_{\lambda,\mu}^{(n)}\left(\pm ct^\lambda\right),\quad \left(\operatorname{Re}(p)>|c|^{\frac{1}{\lambda}}\right) \quad (16)$$

Then Eq. (15) leads to

$$U(\zeta,t)=\sqrt{\frac{2}{\pi}}\frac{1}{\zeta}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\zeta^{2(k+1)}t^{k+1}E_{1-\beta,2+\beta k}^{(k)}\left(-\eta\zeta^2t^{1-\beta}\right)+\sum_{n=0}^{m}\eta b_n(\zeta)$$
$$\times\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\zeta^{2(k+1)}t^{2-\beta}E_{1-\beta,\beta k-\beta+2}^{(k)}\left(-\eta\zeta^2t^{1-\beta}\right) \quad (17)$$

where $E_{\alpha,\beta}^{(k)}(y)=\frac{d^k}{dy^k}E_{\alpha,\beta}(y)=\sum_{k=0}^{\infty}\frac{(j+k)!\,y^j}{j!\,\Gamma(\alpha j+\alpha k+)}$ is the Mittag-Leffler function in two parameters [10].

Now considering the inverse Fourier sine integral transform of Eq. (17). We get

$$u(x,t)=\sqrt{\frac{2}{\pi}}\int_0^\infty\sin(\zeta x)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{1}{\eta}\right)^{k+1}t^{\beta k-1}\left[\sqrt{\frac{2}{\pi}}\frac{1}{\zeta}t^{\beta+1}E_{\beta-1,\beta+k+1}^{(k)}\left(-\frac{1}{\zeta^2\eta}t^{\beta-1}\right)\right.$$
$$\left.+\sum_{n=0}^{m}\eta b_n(\zeta)t^{2n}E_{\beta-1,k+1}^{(k)}\left(-\frac{1}{\zeta^2\eta}t^{\beta-1}\right)\right]d\zeta \quad (18)$$

which is the exact solution of (7).

2.**Spicial Cases:**

Now we consider the following two cases:

Case 1: when $0<\beta\le 1$:

Then equations (7),(8),(9) and (10) leads to

$$\frac{\partial u(x,t)}{\partial t}=\left(1+\eta D_t^\beta\right)\frac{\partial^2u(x,t)}{\partial x^2},\quad 0<\beta\le 1 \quad (19)$$

$$u(x,0)=b_0(x),\ x>0 \quad (20)$$

$$u(0,t)=1,\quad t>0 \quad (21)$$

$$u(x,t)\to 0\quad for\quad x\to\infty \quad (22)$$

Making use the Fourier sine integral transform and boundary conditions (21), (22). Then Eqs. (19), (20) leads to

$$\frac{\partial U(\zeta,t)}{\partial t}=-\zeta^2\left(1+\eta D_t^\beta\right)U(\zeta,t)+\sqrt{\frac{2}{\pi}}\zeta \quad (23)$$

$$U(\zeta,0)=\sqrt{\frac{2}{\pi}}\int_0^\infty\sin(\zeta x)b_0(x)dx=b_0(\zeta) \quad (24)$$

Hence the Laplace transform of Eq. (19)is

$$\widetilde{U}(\zeta,p)=\sqrt{\frac{2}{\pi}}\frac{1}{\zeta}\sum_{k=0}^{\infty}(-1)^k\zeta^{2(k+1)}\frac{p^{-\beta k-\beta-1}}{\left(p^{1-\beta}+\eta\zeta^2\right)^{k+1}}+\eta b_0(\zeta)\sum_{k=0}^{\infty}(-1)^k \times\zeta^{2(k+1)}\frac{p^{-\beta k-1}}{\left(p^{1-\beta}+\eta\zeta^2\right)^{k+1}} \quad (25)$$

Taking the inverse Laplace transform of Eq. (21) , then Eq. (21) leads to

$$U(\zeta,t)=\sqrt{\frac{2}{\pi}}\frac{1}{\zeta}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\zeta^{2(k+1)}t^{k+1}E^{(k)}_{1-\beta,2+\beta k}\left(-\eta\zeta^2t^{1-\beta}\right)+\eta b_0(\zeta) \times\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\zeta^{2(k+1)}t^{2-\beta}E^{(k)}_{1-\beta,\beta k-\beta+2}\left(-\eta\zeta^2t^{1-\beta}\right) \quad (26)$$

Now considering the inverse Fourier sine integral transform of Eq. (22). We get

$$u(x,t)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\sin(\zeta x)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\zeta^{2(k+1)}t\left\{\begin{array}{l}\dfrac{\sqrt{\frac{2}{\pi}}}{\zeta}t^kE^{(k)}_{1-\beta,2+\beta k}\left(-\eta^2\zeta^2t^{1-\beta}\right)\\ +\eta b_0(\zeta)t^{1-\beta}E^{(k)}_{1-\beta,\beta k-\beta+2}\left(-\eta\zeta^2t^{1-\beta}\right)\end{array}\right\}d\zeta \quad (27)$$

which is the special case of equation (18).
Now we will take special cases of case 1:
1. When $b_0(\zeta)=0$, then Eq. (23) yields

$$u(x,t)=\frac{2}{\pi}\int_0^{\infty}\frac{\sin(\zeta x)}{\zeta}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\zeta^{2(k+1)}t^{k+1}E^{(k)}_{1-\beta,2+\beta k}\left(-\eta^2\zeta^2t^{1-\beta}\right)d\zeta \quad (28)$$

which is the result obtained by Fag and others [2].
2. When $b_0(\zeta)=0,\ \beta=1,$ then Eq. (23) becomes

$$u(x,t)=1-\frac{2}{\pi}\int_0^{\infty}\frac{\sin(\zeta x)}{\zeta}\exp\left(\frac{-\zeta^2}{1+\eta\zeta^2}t\right)d\zeta \quad (29)$$

which is the result obtained by Fetacau and Corina [3].
Case2: when $1<\beta\le 2$:

Then equations (7),(8),(9) and (10) leads to

$$\frac{\partial u(x,t)}{\partial t}=\left(1+\eta D_t^{\beta}\right)\frac{\partial^2u(x,t)}{\partial x^2},\quad 1<\beta\le 2 \quad (30)$$

$$u(x,0)=b_0(x),\ x>0 \quad (31)$$

$$u_t(x,0)=b_1(x),\ x>0 \quad (32)$$

$$u(0,t)=1,\quad t>0 \quad (33)$$

$$u(x,t)\to 0,\ \frac{\partial u(x,t)}{\partial x}\to\infty\quad for\quad x\to\infty \quad (34)$$

Making use the Fourier sine integral transform and boundary conditions (29), (30). Then Eqs. (26), (27),(28) leads to

$$\frac{\partial U(\zeta,t)}{\partial t}=-\zeta^2\left(1+\eta D_t^{\beta}\right)U(\zeta,t)+\sqrt{\frac{2}{\pi}}\zeta,\quad 1<\beta\le 2 \quad (35)$$

$$U(\zeta,0)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\sin(\zeta x)b_0(x)dx=b_0(\zeta) \quad (36)$$

$$U_t(\zeta,0)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\sin(\zeta x)b_1(x)dx=b_1(\zeta) \tag{37}$$

The fractional Laplace transform of Eq. (31) subject to the initial conditions (32), (33) is

$$\begin{aligned}\tilde{U}(\zeta,p)&=\frac{\sqrt{\frac{2}{\pi}}\zeta}{p\left(\zeta^2\eta p^{\beta}+p+\zeta^2\right)}+\frac{\zeta^2\eta b_0(\zeta)p^{\beta-1}}{\zeta^2\eta p^{\beta}+p+\zeta^2}+\frac{\zeta^2\eta b_1(\zeta)p^{\beta-2}}{\zeta^2\eta p^{\beta}+p+\zeta^2}\\
&=\sqrt{\frac{2}{\pi}}\frac{1}{\zeta}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{\eta}\right)^{k+1}\frac{p^{-k-2}}{\left(p^{\beta-1}+\frac{1}{\zeta^2\eta}\right)^{k+1}}\\
&+\eta b_0(\zeta)\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{\eta}\right)^{k+1}\frac{p^{\beta-1-k}}{\left(p^{\beta-1}+\frac{1}{\zeta^2\eta}\right)^{k+1}}\\
&+\eta b_1(\zeta)\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{\eta}\right)^{k+1}\frac{p^{(\beta-1)-(k+1)}}{\left(p^{\beta-1}+\frac{1}{\zeta^2\eta}\right)^{k+1}}\end{aligned} \tag{38}$$

Now taking the Laplace inverse integral transform and inverse Fourier sine integral transform respectively to both sides of Eq. (34), we get the exact solution of Eq. (26) as

$$\begin{aligned}u(x,t)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\sin(\zeta x)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{1}{\eta}\right)^{k+1}t^{\beta k-1}\Bigg[\sqrt{\frac{2}{\pi}}\frac{1}{\zeta}t^{\beta+1}E^{(k)}_{\beta-1,\beta+k+1}\left(-\frac{1}{\zeta^2\eta}t^{\beta-1}\right)\\
+\eta b_0(\zeta)E^{(k)}_{\beta-1,k}\left(-\frac{1}{\zeta^2\eta}t^{\beta-1}\right)+\eta b_1(\zeta)t^2E^{(k)}_{\beta-1,k+1}\left(-\frac{1}{\zeta^2\eta}t^{\beta-1}\right)\Bigg]d\zeta\end{aligned} \tag{39}$$

Now we will take special cases of case 2:

1. When $\beta=2$, then Eq. (35) leads to

$$\begin{aligned}u(x,t)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\sin(\zeta x)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{1}{\eta}\right)^{k+1}t^{2k-1}\Bigg[\sqrt{\frac{2}{\pi}}\frac{1}{\zeta}t^{2}E^{(k)}_{1,k+2}\left(-\frac{1}{\zeta^2\eta}t\right)\\
+\eta b_0(\zeta)E^{(k)}_{1,k}\left(-\frac{1}{\zeta^2\eta}t\right)+\eta b_1(\zeta)t^2E^{(k)}_{1,k+1}\left(-\frac{1}{\zeta^2\eta}t\right)\Bigg]d\zeta\end{aligned} \tag{40}$$

2. When $\beta=2,\, b_0(\zeta)=0,\, b_1(\zeta)=0$, then Eq. (35) becomes

$$u(x,t)=\frac{2}{\pi}\int_0^{\infty}\frac{\sin(\zeta x)}{\zeta}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{1}{\eta}\right)^{k+1}t^{2k+1}E^{(k)}_{1,k+2}\left(-\frac{1}{\zeta^2\eta}t\right)d\zeta \tag{41}$$

which is the result obtained by Salim and El-Kahlout [12].

III. Conclusion

This paper has presented some results about Stokes` first problem. Exact solution of this equation is obtained by using the Fourier sine integral transform and integral Laplace transform. The Caputo fractional derivative is considered in Stokes` first problem as time derivative, where the order of the fractional derivative is considered as $m-1<\beta\le m$. Special cases have

been considered in the cases $\beta=1,\ \beta=2$.

References

[1] Erde`lyi A., Magnus W., Oberhettinger F. and Tricomi F. G., Tables of Integral Transforms. Vol. I , McGraw-Hill, New York, 1954.

[2] Fang S. , Wenchang T., Yaohua Z. and Takashi M. , The Rayleigh- Stokes problem for a heated generalized second grade fluid with fractional derivative model, J. Math. Phys. 7(2006) 1072-1080.

[3] Fetacau C. and Corina F., The Rayliegh-Stokes problem for a heated second grade fluids. Int. J. Non-Linear Mech. 37(2002) 011-1015.

[4] Gorenflo R. and Mainardi F.. Fractional calculus: Integral and differential equation of fractional order, in: Carpinteri A, Minardi F. (Eds), Fractals and fractional Calculus in continuum Mechanics, Springer Verlag, Wien and New York, 1997, 223-276. Available from http:// www.fracalmo.org.

[5] Langlands T. A. M., Solution of a modified fractional diffusion equation. Physica A, 367 (2006), 136-144.

[6] Liu F., Anh V. V., Turner I. and Zhuang P., The fractional advection-dispersion equation. J. Appl. Math. Computing 13(2003), 233-245.

[7] Mainardi F. and Pagnini G., The Wright functions as solution of the time-fractional diffusion equation. Appl. Math. Comput. 141 (2003), 51-62.

[8] Mainardi.F and Pagini G., The role of the Fox-Wright functions in fractional sub-diffusion of distributed order. J. Comput Appl. Math. 207 (2007) 245-257.

[9 Mainardi.F, Pagini G. and Gorenflo R., Some aspects of fractional diffusion equations of single and distributed order. Appl. Math. Comput. 187 (2007) 295-305.

[10] Podlubny I., Fractional differential equations. Academic Press, New York, 1999.

[11] Saichev A. and Zaslavsky G., Fractional kinetic equations: solutions and applications. Chaos 7(1997), 753-764.

[12] Salim T. and El-Kahlout A., Solution of fractional order Rayleigh stokes equations, Adv. Theor. Appl. Mech., Vol. 1, 2008, no. 5, 241-254.

[13] Saxena R. K., Mathai A. M. and Haubold H. J., Solutions of certain fractional kinetic equations and a fractional diffusion equation. ArXiv:0704.1916v1[math.CA] 15 Apr 2007

[14] Schneider W. R. and Wyss W., Fractional diffusion equation and wave equations. J. Math. Phys. 30 (1989), 134-144.

[15] Wyss W., The fractional diffusion equation. J. Math. Phys. 27(1986), 2782-2785.

[16] Yu R. and Zhang H., New function of Mittag-Leffler type and its application in the fractional diffusion-wave equation. Chaos, Solutions Fract. (2005), doi: 10. 1016/j. chaos. 2005.08.151.