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Sequence Entropy and the Complexity Sequence Entropy For Z^n Action

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Abstract. *In this paper, we study the complexity of sequence entropy for Z^n actions. After that, we define $C_\alpha(F_\alpha(\tau))$, $h_\alpha(F_\alpha(\tau))$ and the relationships between sequence entropy and complexity sequence entropy. Finally, comparisons between sequence entropy and complexity sequence entropy have been done.*
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I. Introduction and Background

There are a lot of notions characterizing the variety of the behaviours of the individual trajectories of the ergodic dynamical systems. And it is a natural interest to determine a quantity tool dividing the individual trajectories. It's also obvious that this notion must have a connection with an entropy which is considered as a measure of complexity and chaoticness of the dynamical systems in whole [8]. The complexity of finite object was introduced by A.Komogorov and V.Tihomirov in [1] and it was conjectured that for Z actions the complexity coincides with topological entropy [2], [3]. After introducing a notion of complexity of a finite object, due to A.Komogorov [8] many authors tried to give the different variants of these quantity characteristics. T.Kamae gave a definition of determinated trajectory and etc. [7]

As in ergodic theory one of the main tools to study the dynamical behaviour of a topological dynamical system (i.e. a homeomorphism $T: X \to X$ where X is a compact metric space) is to understand its fundamental factors and extensions.

In the category of topological dynamical system, in 1974 Goodman [6] introduced the notion of topological sequence entropy and studied some properties of null systems which are defined as having zero topological sequence entropy for any infinite sequence. It is a natural question whether we have similar characterizations of topological mixing properties using topological sequence entropy.

In [5], [11], the first characterization of topological weak mixing was obtained using sequence entropy. Namely, the authors localized the notion of sequence entropy by defining sequence entropy pairs and proved that a system is topologically weakly mixing if any pair not in the diagonal is a sequence entropy pair. Moreover, they showed that for a minimal system Kushnirenko'statement remains true module an almost one to one extension, i.e. if a minimal system is null, then it is an almost one to one extension of a topological with discrete spectrum [10], [9], [4]. Sequence entropy for a measure was introduced as an isomorphism invariant by Kushnirenko, who used it to distinguish between transformations with the same entropy and spectral invariant. It was also shown that an invertible measure preserving transformation has discrete spectrum if and only if for any sequence the sequence entropy of the system is zero [3].Recently, in[4] the notion of topological mild mixing was introduction.

Let (X, A, μ, T) be an ergodic system. Let $A = \{a_1, a_2, a_3, \dots, a_k\}$ be a finite set of symbols,(alphabet);

$$\Omega = A^{Z^n} = w = \{w_g : w_g \in A, g \in Z\}$$

be the space of configurations with Tychonoff topology, σ be the shift in this configuration space:

Definition 1.1 A topological dynamical system (TDS for short) we mean a pair $(X, T)$where X is a compact metric space (with metric d) and $T: X \to X$ is a homeomorphism. A topological dynamical system (X, T) is a symbolic system on Z^n, X is the σ invariant closed subset of and T is the restriction of σ to X.

Definition 1.2 For an arbitrary finite subset F of Z^n we denote by A^F the set of configuration on F. Every point

$$w^F = (w_g, g \in F)$$

on this set A^F is called a configuration stamp.

An increasing sequence of integers

$$\tau: 0 = \tau(0) < \tau(1) < \cdots < \tau(\mathrm{k}-1)$$

with $k = 1,2, \dots$ is called a window of size k. For $k = 1,2, \dots$ we denote by $\langle k \rangle$ the window of size k such that

$$\langle k \rangle(i) = i \quad (i = 0,1,2, \dots, k-1)$$

Let $\alpha = \alpha_0\alpha_1\alpha_2 \dots$ be an infinite word over a finite A with $cardA \geq 2$, where $cardA$ denotes the number of elements in A. Let τ be a windows of k. Let $\alpha[n+\tau]$ the word $\alpha_{n+\tau(0)}\alpha_{n+\tau(1)} \dots \alpha_{n+\tau(k-1)}$ over of length k. A finite word $\mu_0, \mu_1, \dots, \mu_{k-1}$ is called a τ −factor of α if $\mu_0, \mu_1, \dots, \mu_{k-1} = \alpha[n + \tau]$ for *some* $n = 0,1,2, \dots$. The set of τ −factor of α is denote by $F_\alpha(\tau)$. We also denote $F_\alpha(k) = F_\alpha(\langle k \rangle)$ [7].

II. Sequence Entropy

Let be A a finite set $\#A \geq 2$. Let $N = \{0,1,2,3, \dots\}$ and A^N be the product space. Let $X_n (n \in N)$ be the projection $A^N \to A$ defined by $X_n = \alpha(n)$ for any $\alpha \in A^N$. Let σ be the shift on the space A^N. Let $\tau: 0 = \tau(0) < \tau(1) < \cdots < \tau(k-1), 0$ be an infinite sequence of integers. We define sequence entropy

$$h_\alpha\big(F_\alpha(\tau)\big) = \lim_{k\to\infty} sup \frac{1}{|k|} H\big(X_{\tau(0)}, X_{\tau(1)}, X_{\tau(2)}, \dots, X_{\tau(k-1)}\big).$$

where $X_0, X_1, X_2, \dots$ are considered as random variables on the probability space $(A^N, .)$ and $H(\dots)$ is the Shannon's entropy of random variables.

Corollary 2.1. For $\alpha \in A^N$, assume that

$$\mu_\alpha = w - \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(\alpha)}$$

exits, where δ_x is the unit measure at $x \in A^N$ and the "*w-lim*" implies the weak limit on the space of measures [7].

Let A be an algorithm defined on some subset of a space of all finite $\{0,1\}$ words and taking values in the set of all finite words of A and $l(p)$ be an amount of sings in a $\{0,1\}$ word p:

Now let $C_p(X)$ define comlexity of the configuration space $w \in X$ relatively to the program P as

$$C_p(X) = \lim_{k\to\infty} sup_{w\in X} \frac{1}{|k|} C_p\big(W|_{I_k}\big)$$

where $I_K = \big\{(i_1, i_2, i_3, \dots, i_n) \in Z^N : -k \leq i_j \leq k, j = 1,2,3, \dots, n\big\}$, $|I_k| = (2k+1)^2$.

Let P be such a program that for an arbitrary program P' we have a constant $C(P, P')$ such that for every stamp the inequality

$$C_p(X) \leq C_{p'}(X) + C_p(P, P') \quad [2].$$

III. Complexity of Sequence Entropy

We define the complexity $F_\alpha(\tau)$ in the usual sense and the maximal pattern complexity by $p_\alpha(k)$ as a function on $k \in \{1,2,3, \dots\}$ by

$$p_\alpha(k) = \#\big\{\alpha_{n+\tau(0)}, \alpha_{n+\tau(1)}, \alpha_{n+\tau(2)}, \dots, \alpha_{n+\tau(k-1)} : n: 1,2, \dots\big\},$$
$$p_\propto(\tau) = sup_\tau \# p_\propto(k).$$

Where the "sup" is taken over all windows τ of size k.

Now we define the complexity $C_\alpha\big(F_\alpha(\tau)\big)$ space of configurations with Tychonoff topology:

$$C_\alpha\big(F_\alpha(\tau)\big) = \lim_{k\to\infty} sup \frac{1}{|k|} \big(p_\alpha(\tau)\big).$$

For windows τ and $\tau^{'}$ of size k and $k+1$, respectively, such that $\tau(i) = \tau^{'}(i)$ for $i = 1,2,3, \dots k$,
we call τ' an immediate extension of τ.

Proposition 3.1. For every symbolic system (X, T) and windows τ and τ' of the size k,

$$C_\alpha\big(F_\alpha(\tau')\big) = C_\alpha\big(F_\alpha(\tau)\big)$$

Let α be a recurrent infinite word over a finite set A. For every symbolic system (X, T) and arbitrary optimal programs P_1 and P_2, let us prove the inequality

$$C_\alpha\big(F_\alpha(\tau')\big) \leq C_\alpha\big(F_\alpha(\tau)\big)$$

From the definition of an asymptotically optimal program we have for an arbitrary stamp $p_\alpha(k)$

$$C_\alpha\big(F_\alpha(\tau')\big) = C_\alpha\big(F_\alpha(\tau)\big) + C_\alpha(P_1, P_2)$$

where $C_\alpha(P_1, P_2)$ is a constant. Thus

$$sup_{w\in X} C_\alpha\big(p_\alpha(\tau')\big) \leq sup_{w\in X} C_\alpha\big(p_\alpha(\tau)\big) + C_\alpha(P_1, P_2)$$

and then

$$\frac{1}{|k|} sup_{w\in X} C_\alpha\big(p_\alpha(\tau')\big) \leq \frac{1}{|k|} sup_{w\in X} C_\alpha\big(p_\alpha(\tau)\big) + \frac{1}{|k|} C_\alpha(P_1, P_2)$$

But for every constant $C_\alpha(P_1, P_2)$ we have

$$\lim_{k\to\infty} \frac{1}{(2k+1)} C_\alpha(P_1, P_2) = 0$$

So

$$C_\alpha\big(F_\alpha(\tau')\big) \leq C_\alpha\big(F_\alpha(\tau)\big)$$

Corollary 3.2. For $\alpha \in A^N$, assume that

$$h_\alpha(F_\alpha(\tau)) = \lim_{k\to\infty} sup \frac{1}{|k|} H(X_{\tau(0)}, X_{\tau(1)}, X_{\tau(2)}, \dots, X_{\tau(k-1)})$$

Assume further that the dynamical symbolic system (X, T) has a partially continuous map.
Then, we have

$$C_\alpha(F_\alpha(\tau)) = \lim_{k\to\infty} sup \frac{1}{|k|} (p_\alpha(k)) > 0$$

Proof. Let (X, A, μ, T) be an ergodic system. We consider X_n is a random variables on the probability space(X, T). Let τ be a windows of k. Since the random variable $X_{\tau(0)}, X_{\tau(1)}, X_{\tau(2)}, \dots, X_{\tau(k-1)}$ on the space of words over A of length k has a distribution which is supported by $F_\alpha(\tau)$, we have

$$h(F_\alpha(\tau)) = \lim_{k\to\infty} sup \frac{1}{|k|} H(X_{\tau(0)}, X_{\tau(1)}, X_{\tau(2)}, \dots, X_{\tau(k-1)})$$
$$\leq \lim_{k\to\infty} sup \frac{1}{|k|} \log_2 \# F_\alpha(\tau) \leq \lim_{k\to\infty} sup \frac{1}{|k|} p_\alpha(\tau)$$

There exists an infinite sequence $\tau: 0 = \tau(0) < \tau(1) < \dots < \tau(\mathrm{k}-1), \dots$ such that $h_\alpha(F_\alpha(\tau)) > 0$. Therefore

$$\lim_{k\to\infty} sup \frac{1}{|k|} p_\alpha(k) > 0$$

Theorem 3.3. Let (X, T) be a symbolic dynamical system. Then

$$h(\sigma) = \lim_{k\to\infty} sup \frac{1}{|k|} \log_2 A_k,$$

where $A_k = Card|\{w|_{\mathrm{I_k}} : \mathrm{w} \in \mathrm{X}\}|$ [2].

Theorem 3.4. Let (X, T) be a symbolic dynamical system on $\boldsymbol{Z^n}$ **.** Then

$$C_\alpha(F_\alpha(\tau)) = h_\alpha(F_\alpha(\tau)).$$

Proof. Let the complexity for sequence entropy $C_\alpha(F_\alpha(\tau))$ of the space X be finite and equal to b. So we have

$$\lim_{k\to\infty} \frac{1}{|k|} sup_{w\in X} C_\alpha(p_\alpha(k)) = b.$$

Then let $\varepsilon > 0$ be an arbitrary number. There is some $n_0 \in N$ such that her $k > n_0 \frac{1}{|k|}$

$$sup_{w\in X} C_\alpha(p_\alpha(k)) < b + \varepsilon.$$

So we have

$$sup_{w\in X} C_\alpha(p_\alpha(k)) < (b + \varepsilon)|k|. \quad (1)$$

The inequality shows us that the number of different restrictions of points of X on the $|p_\alpha(k)|$ set is not bigger than $2^{(b+\varepsilon)|k|+1}$.

To prove this, we can write from the definition,

$$\xi: \bigcup_{n=1}^{\infty} \{0,1\}^n \to \bigcup_{\substack{F\subset Z \\ CarF<\infty}} A^F$$

for any program. Now we will find some set U such that

$$U \subset \bigcup_{n=1}^{\infty} \{0,1\}^n \quad and\ \xi(U) = V$$

where

$$V = \{\tau' = (w_g, g \in I_k)\ \exists \tau' \notin X, \tilde{\tau}|_{\mathrm{I_k}} = \tau'\} = A^{I_k} \cap X|_{\mathrm{I_k}}.$$

So we have

$$\#\xi^{-1}(\{A^{I_k} \cap X|_{\mathrm{I_k}}\}) \geq \#(\{A^{I_k} \cap X|_{\mathrm{I_k}}\})$$

Let fix

$$U \subset \bigcup_{n=1}^{sup\,(\tau|\mathrm{I_k})} \{0,1\}^n$$

We will show that

$$\xi(U) \subset A^{I_k} \cap X|_{\mathrm{I_k}}$$

Let us take any

$$\tau' \notin A^{I_k} \cap X|_{\mathrm{I_k}}$$

From the definition $C_\alpha(F_\alpha(\tau'))$ we have $C_\alpha(F_\alpha(\tau')) \leq supC_\alpha(F_\alpha(\tau))$. So there is some finite word $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \in \{0,1\}^n$, $n \leq supC_\alpha(F_\alpha(\tau'))$ such that $\xi\{\alpha_1, \alpha_2, \dots, \alpha_n\} = \tau'$. Thus

$$\xi(U) = A^{I_k} \cap X|_{\mathrm{I_k}}$$

Now we will show that

$$\#U \leq 2^{(b+\varepsilon)|k|}.$$

Indeed, from (1) we have

$$U = \bigcup_{n=1}^{sup\,(\tau|\mathrm{I_k})} \{0,1\}^n \subset \bigcup_{n=1}^{(b+\varepsilon)|k|} \{0,1\}^n = \sum_{n=1}^{(b+\varepsilon)|k|} 2^n = 2^{(b+\varepsilon)|k|+1}.$$

So we have

$$\#V \leq \#U \leq 2^{(b+\varepsilon)|k|+1}$$

From Theorem 3.3 and (1) we have

$$\#(|\{\tau|I_k : \tau \in X_n\}|) \leq 2^{(b+\varepsilon)|k|+1},$$

and then

$$\lim_{k\to\infty} sup \frac{1}{k} \log_2 \# (|\{\tau I_k : \tau \in X_n\}|) \leq \lim_{k\to\infty} sup \frac{1}{k} \log_2 \#^{(b+\varepsilon)|k|+1}$$

$$h(\tau) \leq \frac{1}{k} \log_2 \# (|\{\tau I_k : \tau \in X_n\}|) \leq b + \varepsilon.$$

Hence

$$h_\alpha\big(F_\alpha(\tau)\big) \leq C_\alpha\big(F_\alpha(\tau)\big).$$

Now we will prove the inverse inequality. Let $h_\alpha\big(F_\alpha(\tau)\big) \leq b$. Then for $\varepsilon > 0$ there exists $n_0 \in N$ such that $\forall k > n_0$ we write

$$\frac{1}{|k|} \log_2 \# (|\{\tau|I_k : \tau \in X_n\}|) \leq b + \varepsilon$$

$$\log_2 \#(|\{\tau|I_k : \tau \in X_n\}|) \leq (b+\varepsilon)|k|$$

$$\#(|\{\tau|I_k : \tau \in X_n\}|) \leq 2^{(b+\varepsilon)|k|+1}.$$

Now let us fix some $k > k_0$. For this k we can define some finite program ξ such that it is defined on the finite $\alpha \in \{0,1\}^{(b+\varepsilon)|k|+1}$ and give us all the finite restriction of the space X on I_k. Now we will continue with the program ξ in the following way.

One will divide the big cube I_{km} into $\frac{|I_{km}|}{|I_k|}$ domains every part of which is equal to I_k and now consider the program ξ on each domain of the big cube. Certainly this program ξ will be defined on the $\{0,1\}$ words of length not bigger than

$$(b+\varepsilon)k \frac{|I_{km}|}{|I_k|} = (b+\varepsilon)|k_{km}|\,,$$

thus the complexity of the space X relatively to this program ξ is not bigger than $(b+\varepsilon)$. Because of that the complexity of an arbitrary asymptotically optimal program ξ will not be bigger than b.

The proof is complete.

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