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A High Order Continuation Based On Time Power Series Expansion And Time Rational Representation For Solving Nonlinear Structural Dynamic Problems

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Abstract: In this paper, we propose a high order continuation based on time power series expansion and time rational representation called Padé approximants for solving nonlinear structural dynamic problems. The solution of the discretized nonlinear structural dynamic problems, by finite elements method, is sought in the form of a power series expansion with respect to time. The Padé approximants technique is introduced to improve the validity range of power series expansion. The whole solution is built branch by branch using the continuation method. To illustrate the performance of this proposed high order continuation, we give some numerical comparisons on an example of forced nonlinear vibration of an elastic beam.

Key words: Nonlinear structural dynamics, finite element, power series expansion, Padé approximants, high order continuation, nonlinear forced vibration beam.

I. Introduction

During the last two decades, several research activities have been devoted to the development of efficient algorithms for accessing numerically to the response of a nonlinear elastic structure subjected to dynamic loading. In literature, there is a variety of numerical approaches for solving this type of nonlinear dynamic problems. Some of them are currently implemented in several computer codes. The most popular ones are of iterative or prediction-correction type [1, 2, 3, 4, 5].

These methods are generally based on direct integrations of the equations that govern the studied nonlinear dynamic problems and which employ procedures for spatial classical discretization by finite elements, finite differences, finite volumemethods.... The time integration is performed by an implicit or explicit time scheme. These methods work very well and allow to obtain the nonlinear dynamic response, but this might require a very important CPU time of computation especially for implicit algorithms.

The implicit schemes are unconditionally stable and allow more freedom on the choice of time step size, which will be only limited by the accuracy requirements [3, 6]. Due to the large size of media meshes, the implicit schemes require large memory as well as many computations. For example, the Newmark implicit time schemes are unconditionally stable if the Newmark parameters are: β and γ satisfy $\frac{1}{2} \leq \gamma \leq 2\beta$ [6]. As for explicit schemes, they are conditionally stable. In many situations this forces the time step size to be very small and causes an increase in the number of time steps required for the integration. The explicit scheme can lead to stability problems related to time step size, when the structural mesh is much smaller than that for the media mesh.

These numerical methods permit to solve the considered nonlinear structural dynamic problem in a step wise fashion and a full point by point description is obtained. Previous research reported that the perturbative methods coupled with implicit time schemes are well adapted to the resolution of these of nonlinear structural dynamic problems. These high order implicit algorithms are variants of the Asymptotic Numerical Method (ANM) [7]. They are derived from associating homotopical transformation, space time discretisation procedures and the perturbative analysis. The adopted parameter perturbation is artificially introduced via the homotopical transformation. A key point of these implicit high order algorithms is the possibility to choose an operator and to compute many time steps with a single matrix triangulation. Other perturbation analyses using the time variable as perturbation parameter are firstly proposed in [8, 9]. The Qaisi's work has been devoted to study the nonlinear vibration of clamped-supported beams. The dynamic response is sought in the form of a double series with respect to spatial and temporal variables introducing a time scale varying in $[-1,+1]$ and that oscillates harmonically. To investigate the linear dynamic problems having a small number degrees of freedom, Fafard and al. have applied a time perturbation method to obtain the linear dynamic response of the structure. This alternative has been also extended to multi degrees of freedom linear dynamic problems.

The aim of this work is to extend this idea of time series expansion to nonlinear structural dynamic problems with multidegrees of freedom, and secondly to introduce other fractional representations based on the Padé approximants [12] in order to ameliorate the validity range of the considered time power series expansion. The whole nonlinear dynamic response of the structure is obtained by continuation method [12]. The efficiency of the proposed high order continuation is illustrated on an example of nonlinear forced vibration beam. A comparison with Newton Raphson method coupled with Newmark implicit scheme [6] is presented.

II. Governing nonlinear structural dynamic equations

Let's consider a tridimensional nonlinear elastic structure occupying a volume Ω of boundary $\partial\Omega$ The structure, made of an isotropic homogeneous elastic material of density ρ, is subjected to a prescribed displacement u_d on a part $\partial\Omega_u$ and to a density of surface forces F on the complementary $\partial\Omega_F$. Moreover, is submitted to a density of body forces f (see figure 1).

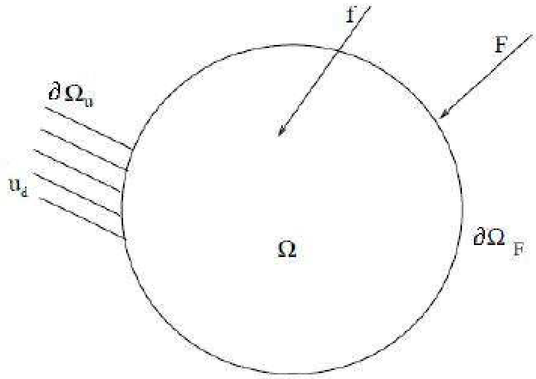

***Figure 1**. Three-dimensional elastic structure*

Neglecting the damping forces and applying the virtual work principle, the dynamic motion of the structure is given by the variational formulation:

$$\begin{cases} \int_\Omega \rho \frac{\partial^2 u}{\partial t^2} \delta u d\Omega + \int_\Omega S : \delta\gamma d\Omega = \int_\Omega f \delta u d\Omega + \int_{\partial\Omega_F} F \delta u ds \\ u(x, t = t_0) = u_0(x) \\ \frac{\partial u}{\partial t}(x, t = t_0) = \dot{u}_0(x) \\ u(x,t) = u_d(t) \qquad on \qquad x \in \partial\Omega_u \end{cases} \tag{2.1}$$

where u is the displacement vector, $u_0(x)$, $\dot{u}_0(x)$ and $u_d(t)$ are respectively the initial and boundary conditions, $\gamma(u)$ is the nonlinear Green-Lagrange strain tensor defined by :

$$\gamma(u) = \gamma_L(u) + \gamma_{NL}(u, u) \tag{2.2}$$

with $\gamma_L(u)$ and $\gamma_{NL}(u, u)$ are respectively the linear and nonlinear quadratic parts of strain tensor, S is the Piola-Kirchhoff stress tensor connected to $\gamma(u)$ via the constitutive law:

$$S = D : \gamma(u) \tag{2.3}$$

with D is the elastic tensor. The aim of this work is to propose a high order continuation for solving the nonlinear dynamic problem governed by the equations (2.1), (2.3). This algorithm is developed using finite elements method and the power series expansion and rational representation called Padé approximants with respect to time variable t. The proposed algorithm is presented and tested in the following sections.

III. The proposed algorithm

To solve the variational problem (2.1), (2.3), we develop, in this section, a high order continuation based on the following three steps:

a- Space discretization by finite elements method of the nonlinear equations (2.1), (2.3)

b- The solution of these discretized equations is sought in the form of a power series expansion with respect to time t. This power series representation is improved by the so-called Padé approximants [12] c- Application of the continuation method to build the whole solution branch by branch [12].

3.1. Space discretization by finite elements method

The space discretization of the equations (2.1) and (2.3) by the finite elements method [1, 2, 13] is obtained in the following form:

$$\begin{cases} \sum_e \int_{\Omega^e} \rho \ {}^tNNd\Omega^e + \sum_e \int_{\Omega^e} \ {}^tB(q)Sd\Omega^e = F \\ S = C\gamma = D\left(B_L + \frac{1}{2}B_{NL}(q)\right)q \\ q(t = t_0) = q_0 \\ \dot{q}(t = t_0) = \dot{q}_0 \\ q = q_d \qquad on \qquad \partial\Omega_u \end{cases} \tag{3.1}$$

where q and $\ddot{q}$ are respectively the nodal displacement and acceleration vectors, q_0, $\dot{q}_0$ and $\dot{q}_0$ are respectively the discretized initial and boundary conditions, N is the shape matrix such that $u = Nq$ and $B = B_L + B_{NL}(q)$ with B_L, B_{NL} are the linear and nonlinear strain matrix corresponding respectively to the discretization of the linear and nonlinear parts of γ, C is the matrix of elasticity and F is the space discretized forces vector.

Using the usual notations, the expression (3.1-a) can be written in the matrix form:

$$M\ddot{q} + K(q)q = F \tag{3.2}$$

where

$$\begin{cases} M = \sum_e \int_{\Omega^e} \rho \ {}^tNNd\Omega^e \\ K(q) = \sum_e \int_{\Omega^e} \ {}^tB(q)(B_L + \frac{1}{2}B_{NL}(q))d\Omega^e \end{cases} \tag{3.3}$$

are the mass and stiffness matrices. The time integration of the nonlinear problem (3.2) completed by (3.1-b) and (3.1-c) is generally carried out by classical numerical methods which use explicit or implicit time schemes [2, 3, 14]. We present, in the following section, another alternative for solving the nonlinear system (3.2), (3.1-b) and (3.1-c). It is based on the power series expansion with respect to time t.

3.2. Power series expansion technique

The structural dynamic response q(t) of nonlinear problem (3.2), (3.1-b) and (3.1-c) is sought in the form of a power series expansion with respect to time $(t - t_0)$ truncated at the order p:

$$q(t) = q_0 + (t - t_0)q_1 + (t - t_0)^2q_2 + \cdots + +(t - t_0)^pq_p, \qquad t \in [\![0, t_{maxs}]\!] \tag{3.4}$$

where the unknown vector q_j is time independent, t_{maxs} is the validity range of the series (3.4), see [15] and t_0 is the initial time. Then, we expand the stress vector S with respect to $(t - t_0)$ in the form:

$$\mathrm{S}(t) = \sum_{j=0}^{p}(t - t_0)^jS_j\,, \qquad t \in [0, t_{maxs}] \tag{3.5}$$

and we assume, in addition, that the vector force $F(t)$, in the problem (3.2), is developable in time up to the order $(p - 2)$ as:

$$\mathrm{F}(t) = \sum_{j=0}^{p}(t - t_0)^jF_j \tag{3.6}$$

By substitution of (3.4), (3.5) and (3.6) in (3.2), (3.1-b), we get the terms q_j and S_j $(0 \le j \le p)$ of time series (3.4) and (3.5) are given by the following expressions:
For $j = 0$:

$$\begin{cases} q_0 = u_0 \\ S_0 = D(B_L + \frac{1}{2}B_{NL}(q_0))q_0 \end{cases} \tag{3.7}$$

For $j = 1$:

$$\begin{cases} q_1 = \dot{u}_0 \\ S_1 = D(B_L + \frac{1}{2} B_{NL}(q_0))q_1 \end{cases} \quad (3.8)$$

For $2 \leq j \leq p$:

$$\begin{cases} q_j = \frac{1}{j(j-1)} M^{-1}(F_{j-2} - \sum_e \int_{\Omega^e} B_L S_{j-2} d\Omega^e - \sum_{i=0}^{i=j-2} \sum_e \int_{\Omega^e} B_{NL}(q_{j-i-2}) S_i d\Omega^e) \\ S_j = D(B_L q_j + \sum_{i=0}^{i=j} B_{NL}(q_i)) q_{j-i} \end{cases} \quad (3.9)$$

The expressions (3.7), (3.8) and (3.9) give each term of the series (3.4) and (3.5) in function of those computed at previous orders by inverting only one mass matrix M. They are fully determined and the solution is computed at any time in the interval $[0, t_{maxs}]$. It is important to underline that the validity range $[0, t_{maxs}]$ of the series representation (3.4) is finite and must be computed.

3.3. Validity range of the power series

When the power series expansions (3.4) and (3.5) are truncated at a given order p, it is valid only up to a certain maximal time t_{maxs} [7, 12], given by the criterion:

$$t_{maxs} = \left(\kappa \frac{\|q_1\|}{\|q_p\|}\right)^{\frac{1}{p-1}} \quad (3.10)$$

which requires that the power series representation ceases to be valid when the difference between the power series at two consecutive orders is small than a tolerance parameter _. This way gives the validity range that depends on the truncate order p and on the required tolerance. The nonlinear dynamic structural response is computed branch by branch by using the continuation method [15].

3.4. Improvement of validity range by Padé approximants

It is possible to improve the critical time t_{maxs} by replacing the power series expansion (3.4) and (3.5) by a rational representation one called Padé approximants [15, 16, 17]. This rational representation of the solution paths $q(t)$ and $S(t)$ is given in the form:

$$\begin{cases} P_p^q(t) = q_0 + (t - t_0) \frac{D_{p-2}}{D_{p-1}} q_1 + \cdots + (t - t_0)^p \frac{1}{D_{p-1}} q_p, & t \in [0, t_{maxp}] \\ P_p^S(t) = S_0 + (t - t_0) \frac{D_{p-2}}{D_{p-1}} S_1 + \cdots + (t - t_0)^p \frac{1}{D_{p-1}} S_p, & t \in [0, t_{maxp}] \end{cases} \quad (3.11)$$

where $D_j(t)$ are polynomials of degree j with real coefficients $(d_j)\{j = 1, p - 1\}$:

$$D_j(t) = 1 + td_1 + t^2 d_2 + \cdots + t^j q_j \quad (3.12)$$

with d_j are calculated as in [12]. Those rational representations have been tested in many cases [18]. The Padé approximants improve significantly the validity range of the power series expansion (3.4) and (3.5). To get the critical time t_{maxp} of the rational representation (3.11), one has only to require that the difference between two rational representations at consecutive orders remains small at the end of the step. The maximal value t_{maxp} is then defined by [12]:

$$\frac{\left\|P_p^q(t_{maxp}) - P_{p-1}^q(t_{maxp})\right\|}{\left\|P_p^q(t_{maxp}) - q_0\right\|} < \delta \quad (3.13)$$

where δ is a tolerence parameter. The solution is obtained branch by branch using the rational representation (3.11) and the relation (3.13) [12].

3.5. Continuation method

As soon as the critical times $t = t_{maxs}$ or $t = t_{maxp}$ evaluated respectively by the criteria (3.10) or (3.13), the complete solution is obtained branch by branch by means of the continuation method [12, 15]. At the end of each branch, we start with the new initial conditions computed at critical time t_{maxs} or t_{maxp} of the previous step.

IV. Application: Forced nonlinear vibrating 2D elastic beam

We apply, in this paragraph, the proposed high order continuation in order to test its efficiency for solving the forced nonlinear vibration problem of a bidimensional elastic beam. We consider a $2D$ beam, of length $L = 200mm$ and of width $l = 10mm$, which is made of an homogeneous, isotropic and elastic material with density $\rho = 10^{-4}Kgm^{-3}$, Young modulus $E = 10^5 MPa$ and Poisson's ratio $\upsilon = 0.3$. The structure is clamped at one side and subjected to the other side to an harmonic force $F = -100cos(10000t)$. The initial values are assumed to be equal to zero. The beam is meshed in 12 quadriangle $Q8$ elements (see figure 2).

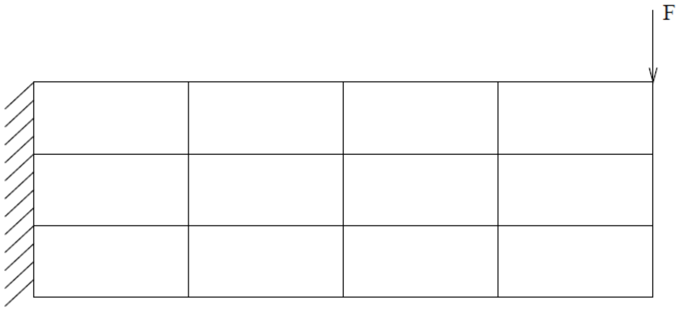

Figure 2. *Beam meshed in 12 quadriangle $Q8$ elements subjected to an harmonic force F*

The numerical results obtained, in time interval $[0, 0.001s]$, by the high order continuation are compared to those computed by the Newton Raphson method using the Newmark implicit time scheme which we will call next as a reference solution. The choice of this time interval is due to the smallest period of the excitation force F. The Newmark parameters and time step adopted in the numerical computation are: $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$, $\Delta t = 10^{-6}s$. We have verified that the optimal timestep corresponds to $\Delta t = 10^{-6}s$, gives a sufficient accuracy. For greater Δt, the Newton Raphson solution diverges. We note that the reference solution, in the considered temporal domain, is get in 2000 iterations. In the following, we present the transversal displacement at loaded node versus time t. The qualities of solutions are characterized by the logarithmof the norm of residual vectors. This residual vector, denoted by R, is defined by:

$$R = M\ddot{q} + K(q)q - F \tag{4.1}$$

4.1. Power series solution without continuation method

The comparison of power series solutions (see equation (3.4)), truncated at orders 10, 20, 30, 40, 50, 60 and 70, without continuation method with the Newton Raphson solution is reported in figure 3.

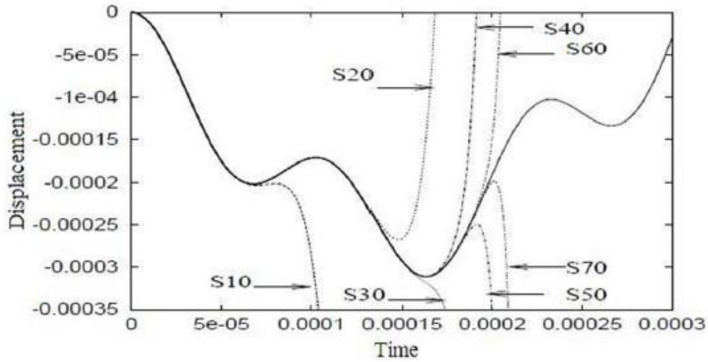

Figure 3. *Power series solutions (S10, S20, S30, S40, S50, S60, S70), truncated at orders 10, 20,30, 40, 50, 60 and 70, compared to Newton Raphson reference solution*

The power series solutions are denoted by $S10, S20, S30, S40, S50, S60$ and $S70$ (dashed curves) and the curve of the Newton Raphson solution is plotted with full line. This figure shows that the power series solutions are confounded with Newton Raphson solution in the interval $[0, t_{maxs}]$. This power series representation (see equation (3.4)) is only valid up to critical time t_{maxs}. For higher truncation order (order 70), the power series solution seems to be valid until a critical time $t_{maxs} = 2.10^{-4}$. The critical time t_{maxs} increases with the truncation order. Then, by increasing the truncation order, we ameliorate the quality of the solution. The qualities of the power series solutions are better when we increase the truncate order p. For large p greater than 50, they become more and more weak. The effect of the tolerance parameter κ and of the truncation order p on the critical time t_{maxs}, without continuation process, is given in the table 1.

p	$\kappa = 10^{-8}$	$\kappa = 10^{-10}$	$\kappa = 10^{-12}$	$\kappa = 10^{-14}$
10	1.08569221 10^{-5}	0.610529597 10^{-5}	0.343326022 10^{-5}	0.19306641 10^{-5}
20	6.38350945 10^{-5}	4.94251953 10^{-5}	0.382681338 10^{-5}	2.96296262 10^{-5}
30	10.1652124 10^{-5}	8.62358577 10^{-5}	7.31575775 10^{-5}	6.20627113 10^{-5}
40	12.3932471 10^{-5}	10.978766 10^{-5}	9.7257242 10^{-5}	8.61569608 10^{-5}
50	14.3472306 10^{-5}	13.0347112 10^{-5}	11.8422642 10^{-5}	10.7589052 10^{-5}
60	15.5696673 10^{-5}	14.3812479 10^{-5}	13.2835396 10^{-5}	12.2696184 10^{-5}
70	16.5086596 10^{-5}	15.4276596 10^{-5}	14.4174443 10^{-5}	13.4733788 10^{-5}
80	17.3235804 10^{-5}	16.3303928 10^{-5}	15.3941461 10^{-5}	14.511576 10^{-5}
90	17.9096622 10^{-5}	16.9965244 10^{-5}	16.1299436 10^{-5}	15.3075461 10^{-5}
100	18.4172502 10^{-5}	17.5718151 10^{-5}	16.7651893 10^{-5}	15.9955913 10^{-5}
110	18.8477023 10^{-5}	18.0609211 10^{-5}	17.3069834 10^{-5}	16.5845181 10^{-5}
120	19.1967345 10^{-5}	18.461977 10^{-5}	17.7553424 10^{-5}	17.0757544 10^{-5}

Table 1. *Influence of the tolerance parameter κ and of the truncation order p on the critical time t_{maxs} without continuation method.*

One can see that the critical time t_{maxs} increases, for a fixed tolerance κ, with the truncation order and decreases, for a fixed order p, with the tolerance parameter κ.

4.2. Complete solution

In the table 2, we report the influence of the tolerance parameter κ and the truncation order p on the number of steps necessary to obtain the complete solution on the time interval $[0,0.001s]$.The symbols $NPAS$ and SM, in the table 2, denote respectively the number of continuation steps and the number of computed right hand sides such that : $SM = NPAS(p - 2)$. Optimal orders are those mentioned in bold in table 2.

p	$\kappa = 10^{-8}$		$\kappa = 10^{-10}$		$\kappa = 10^{-12}$		$\kappa = 10^{-14}$	
	NPAS	SM	NPAS	SM	NPAS	SM	NPAS	SM
10	75	600	125	1000	209	1672	351	2808
20	**16**	**288**	20	360	26	468	33	594
30	10	280	**12**	**336**	**14**	**392**	**16**	**448**
40	9	342	10	380	11	418	12	456
50	7	336	8	384	9	432	10	480
60	7	406	7	406	8	544	9	522
70	7	476	7	476	7	476	8	544
80	6	468	7	546	7	546	7	546
90	6	528	6	528	7	616	7	616
100	6	588	6	588	7	686	7	686
110	6	648	6	648	6	648	7	756
120	6	708	6	708	6	708	6	708

Table 2. *Influence of the tolerance parameter κ and of the truncate order p on the number continuation steps*

Fixing the time interval $[0,0.001s]$ and for a given tolerance parameter $\kappa = 10^{-10}$, for example, the optimal truncation order for the continuation method is $p = 30$, it corresponds to a minimal number of computed right hand sides. The complete solution computed by the high order continuation, denoted by $(S30c)$, using the power series truncated at order $p = 30$ with $\kappa = 10^{-10}$ compared to the Newton Raphson solution denoted by (N) is plotted in figure 4.

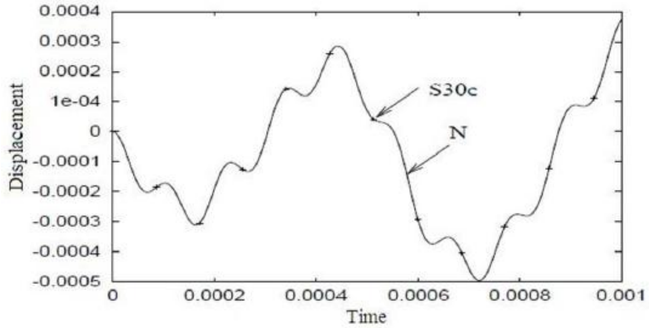

Figure 4*. Comparison between the reference solution and the power series solution truncated at order 30 with continuation method,* $\kappa = 10^{-10}$.

The proposed high order continuation gives the same result as the Newton Raphson method as shown in figure 4. The complete solution is obtained in 12 continuation steps, with the parameters $p = 30$ and $\kappa = 10^{-10}$. This solution is obtained by computing 336 right hand sides and inverting only one mass matrix. While, the Newton Raphson method gives the same result using 2000 iterations, an inversion of tangent matrix at each iteration is required 2000 triangulations, to obtain the complete solution.

4.3. Padé approximants solution without continuation method

The Padé series solutions (see equation (3.11)) denoted respectively by $P10, P20, P30$ and $P40$, truncated at respective orders 10, 20, 30 and 40 without continuation method compared to Newton Raphson reference solution (N) are plotted in figure 5 with $\kappa = 10^{-10}$.

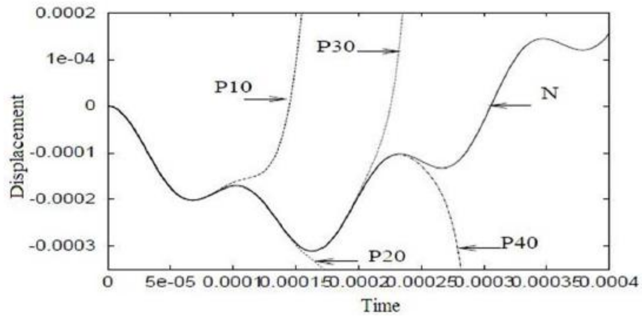

Figure 5*. Comparison between the Newton Raphson reference solution (N) and Padé approximants solution (P10, P20, P30 and P40) without continuation method truncated at respective orders 10, 20, 30 and 40*

As for the power series solution, the validity range of the time Padé solution increases with truncature order. The qualities of these four Padé approximants solutions denoted respectively by $rP10, rP20, rP30$ and $rP40$ without continuation method compared to that (rN) computed by the Newton Raphson method are reported in figure 6.

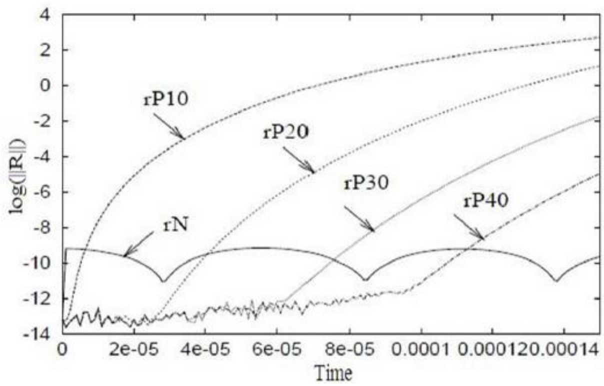

Figure 6*. Comparison of qualities of Padé approximants solutions (rP10, rP20, rP30 and rP40) truncated at respective orders 10, 20, 30 and 40 without continuation method and the Newton Raphson reference solution (rN)*

One can see, on figure 7, that the use of Padé approximants ameliorates substantially the validity range of the power series solution. The three qualities of these solutions are given in the figure 8. Let us remark that the power series solutions and Padé approximants have the same quality in the interval $[0, 5.10^{-5}s]$. For $t = 5.510^{-5}s$, the quality of Padé representation truncated at order 30 seems to be better than power series expansion. For very small time $t = 5.510^{-5}s$, the power series expansion and its improvement by Padé approximant are more accurate than the Newton Raphson one.

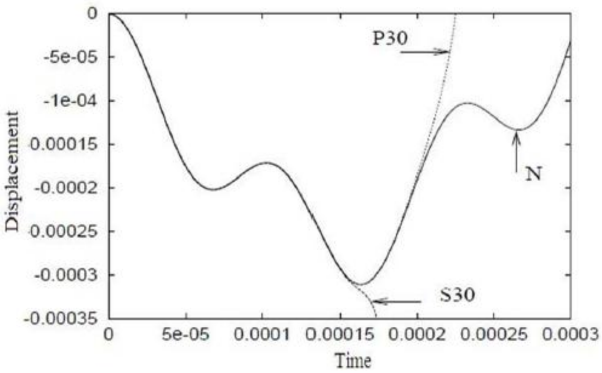

Figure 7. *Comparison between the Padé approximants solution (P30), power series solution (S 30) truncated at orders 30 without continuation method and the Newton Raphson reference solution (N)*

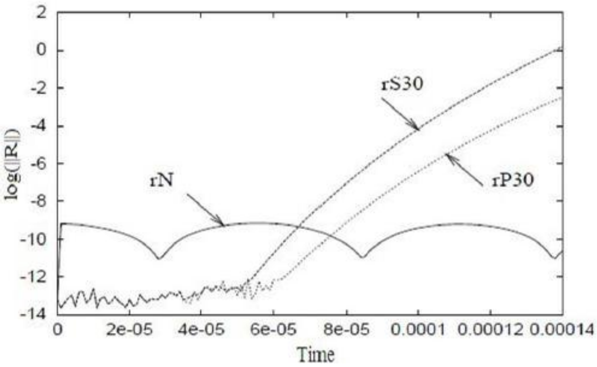

Figure 8. *Comparison of the qualities of the Pad´e approximants solution (rP30), power series solution (rS 30) truncated at orders 30 without continuation method and the Newton Raphson reference solution (rN)*

4.4. Padé approximants solution with continuation method

The complete solution obtained by the Padé approximants solution truncated at order 30 using the continuation method is reported in figure 9 and compared to those obtained by power series and reference solutions.

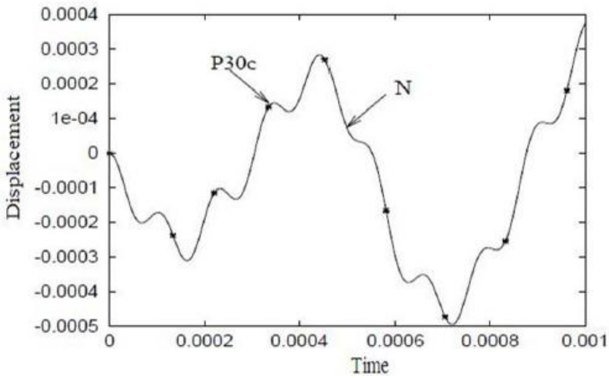

Figure 9. *Comparison of three solutions: Padé approximants (P30c), power series (S 30c) truncated at order 30 with continuation process and reference solution (N),* $\kappa = 10^{-10}$, $\delta = 10^{-6}$.

With the Padé approximants, this solution path requires only 9 steps, whereas the same solution requires 12 steps with using the power series expansion. So, it appears that the use of the Padé approximants leads to a reduction of the computational steps. The Comparison of the qualities of the Padé approximants solution ($rP30$), power series solution ($S30c$) truncated at orders 30 and the Newton Raphson solution (rN) are represented in figure 10.

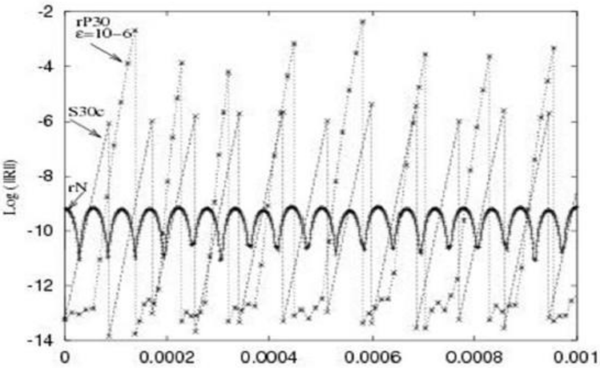

Figure 10. *Comparison of the qualities of the Padé approximants solution (rP30), power series solution (S 30c) truncated at orders 30 with continuation method and the Newton Raphson reference solution (rN)*

This example shows that the Padé approximants combines efficiency and robustness. Indeed, it reduced the number of steps with respect to the power series and it permitted to compute this vibration problem without any difficulty.

V. Conclusion

The paper has described an algorithm for solving nonlinear structural dynamic problems. It is based on the combination of a classical finite elements discretization procedure, a time power series expansion and time rational representation called Padé approximants and a continuation method. The different terms of this asymptotic expansion are obtained in a simple recurrent way. Their numerical computation requires only a single inversion of the mass matrix. The introduction of Padé approximants permits to improve the validity range of the power series solutions. The effectiveness of the proposed algorithm is tested in a problem of a forced nonlinear vibration of an elastic beam. The determination of the whole solutions improved by Padé approximants is performed by means of a continuation method.

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