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Fekete-Szegö Problems for Quasi-Subordination Classes

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Abstract : An analytic function f is quasi-subordinate to an analytic function g, in the open unit disk if there exist analytic functions φ and w, with $|\varphi| \le 1$, w(0) = 0 and |w(z)| < 1 such that $f(z) = \varphi(z)g(w(z))$. Certain subclasses of analytic univalent functions associated with quasi-subordination are defined and the bounds for the Fekete-Szegö Coefficient functional $|a_3 - \mu a_2^2|$ for functions belonging to these subclasses are derived.

Keywords: Fekete-Szegö functional, analytic function, subordination, quasi-subordination, univalent function.

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I. Introduction

Let A be the class of analytic function f in the open unit D = {z : |z| < 1} normalized by f(0) = 0 and $f'(0) = 1$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. For two analytic functions f and g, the function f is subordinate to g, written as follows:

$$f(z) \prec g(z) \qquad (1.1)$$

if there exists an analytic function w, with w(0) = 0 and |w(z)| < 1 such that f(z) = g(w(z)). In particular, if the function g is univalent in D, then $f(z) \prec g(z)$ is equivalent to f(0) = g(0) and $f(D) \subset g(D)$. For brief survey on the concept of subordination, see [1].

Ma and Minda [2] introduced the following class

$$S^*(\phi) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\}, \qquad (1.2)$$

where ϕ is an analytic function with positive real part in D, $\phi(D)$ is symmetric with respect to the real axis and starlike with respect to $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in S^*(\phi)$ is called Ma-Minda starlike (with respect to ϕ). The class C (ϕ) is the class of functions f ∈ A for which $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$. The class $S^*(\phi)$ and $C(\phi)$ include several well-known subclasses of starlike and convex functions as special case.

For two analytic functions f and g, the function f is quasi-subordinate to g, written as follows:

$$f(z) \prec_q g(z) \qquad (1.3)$$

if there exist analytic functions φ and w, with $|\varphi(z)| \le 1$, w(0) = 0 and |w(z)| < 1 such that $f(z) = \varphi(z)g(w(z))$. Observe that $\varphi(z) = 1$, then f(z) = g(w(z)), so that $f(z) \prec g(z)$ in D. Also notice that if w(z) = z, then $f(z) = \varphi(z)g(z)$ and it is said that f is majorized by g and written by f(z) << g(z) in D. Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization.

Throughout this paper it is assumed that ϕ is analytic in D with $\phi(0) = 1$. Motivated by [2,3,28], we define the following classes.

Definition 1.1. Let the class $N_q(\alpha, \phi)$ consists of functions $f \in A$ satisfying the quasi-subordination

$$\frac{\alpha z^2 f''(z) + z f'(z)}{f(z)} - 1 \prec_q \phi(z) - 1 \qquad (1.4)$$

Example 1.1. The function $f : D \to C$ defined by the following

$$\alpha z^2 \frac{f''(z)}{f(z)} + z\frac{f'(z)}{f(z)} - 1 = z(\phi(z) - 1) \qquad (1.5)$$

belongs to the class $N_q(\alpha, \phi)$.

Definition 1.2. Let the class $M_q(\alpha, \lambda, \phi)$, $(\alpha \geq 0)$ consist of functions $f \in A$ satisfying the quasi-subordination

$$z\frac{f'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left[1 + \frac{zf''(z)}{f(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{f'(z)}{f(z)} - 1\right)\right] - 1 \prec_q \phi(z) - 1 \qquad (1.6)$$

Example 1.2. The function $f : D \to C$ defined by the following

$$z\frac{f'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left[1 + \frac{zf''(z)}{f(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{f'(z)}{f(z)} - 1\right)\right] - 1 = z(\phi(z) - 1) \qquad (1.7)$$

belongs to the class $M_q(\alpha, \lambda, \phi)$.

It is well known (see [10]) that the n-th coefficient of a univalent function $f \in A$ is bounded by n. The bounds for coefficient give information about various geometric properties of the function. In this paper, we obtain coefficient estimates for the functions in the above defined classes.

Let Ω be the class of analytic functions w, normalized by $w(0) = 0$ and satisfying the condition $|w(z)| < 1$. We need the following lemma to prove our results.

Lemma 1.1. (see [26]) If $w \in \Omega$, then for any complex number t

$$\left|w_2 - t w_1^2\right| \leq \max\{1, |t|\} \qquad (1.8)$$

The result is sharp for the functions $w(z) = z^2$ or $w(z) = z$.

2. Main Results

Throughout the paper, $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, $\underline{\varphi}(z) = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + \cdots$, $B_1 \in R$ and $B_1 > 0$.

Theorem 2.1. If $f \in A$ belongs to $N_q(\alpha, \phi)$, $\alpha \geq 0$, then

$$|a_2| \leq \frac{B_1}{1+2\alpha}, \quad |a_3| \leq \frac{B_1}{2(1+3\alpha)}\left[1 + \max\left\{1, \frac{B_1}{1+2\alpha} + \left|\frac{B_2}{B_1}\right|\right\}\right] \qquad (2.1)$$

and for any complex number μ,

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1}{2(1+3\alpha)}\left(1 + \max\left\{1, \frac{B_1}{1+2\alpha}\left|1 - \frac{2\mu(1+3\alpha)}{(1+2\alpha)}\right| + \left|\frac{B_2}{B_1}\right|\right\}\right) \qquad (2.2)$$

Proof. If $f \in N_q(\alpha, \phi)$, then there exist analytic functions $\underline{\varphi}$ and w, with $|\underline{\varphi}| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{\alpha z^2 f''(z) + zf'(z)}{f(z)} - 1 = \underline{\varphi}(z)(\phi(w(z)) - 1) \qquad (2.3)$$

Since

$$\frac{\alpha z^2 f''(z) + zf'(z)}{f(z)} - 1 = a_2(1+2\alpha)z + (2a_3(1+3\alpha) - a_2^2(1+2\alpha))z^2 + \cdots \qquad (2.4)$$

$$\phi(w(z)) - 1 = B_1 w_1 z + (B_1 w_2 + B_2 w_1^2)z^2 + \cdots$$

$$\underline{\varphi}(z)(\phi(w(z)) - 1) = B_1 C_0 w_1 z + (B_1 C_1 w_1 + C_0(B_1 w_2 + B_2 w_1^2)z^2 + \ldots \qquad (2.5)$$

it follows from (2.3) that

$$a_2 = \frac{B_1 C_0 w_1}{1+2\alpha}, \quad a_3 = \frac{1}{2(1+3\alpha)}\left[B_1 C_1 w_1 + B_1 C_0\left(w_2 + \left(\frac{B_1 C_0}{(1+2\alpha)} + \left|\frac{B_2}{B_1}\right|\right)w_1^2\right)\right] \qquad (2.6)$$

Since $\underline{\varphi}(z)$ is analytic and bounded in D, we have [27, page 172]

$$|c_n| \le 1 - |c_0|^2 \le 1 \quad (n > 0) \qquad (2.7)$$

By using this fact and the well known inequality, $|w_1| \le 1$, we get

$$|a_2| \le \frac{B_1}{(1+2\alpha)} \qquad (2.8)$$

Further,

$$a_3 - \mu a_2^2 = \frac{1}{2(1+3\alpha)}\left\{B_1 C_1 w_1 + B_1 C_0\left[w_2 + \left(\frac{B_1 C_0}{(1+2\alpha)} + \frac{B_2}{B_1} - \frac{2\mu(1+3\alpha)}{(1+2\alpha)^2} B_1 C_0\right)w_1^2\right]\right\} \qquad (2.9)$$

Then

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{2(1+3\alpha)}\left\{|B_1 C_1 w_1| + \left|B_1 C_0\left[w_2 - \left(\frac{2\mu(1+3\alpha)}{(1+2\alpha)^2} B_1 C_0 - \frac{B_1 C_0}{(1+2\alpha)} - \frac{B_2}{B_1}\right)w_1^2\right]\right|\right\} \qquad (2.10)$$

Again applying $|C_n| \le 1$ and $|w_1| \le 1$, we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1}{2(1+3\alpha)}\left[1 + \left|w_2 - \left(-\left(\frac{1}{(1+2\alpha)} - \frac{2\mu(1+3\alpha)}{(1+2\alpha)^2}\right)B_1 C_0 - \frac{B_2}{B_1}\right)w_1^2\right|\right]$$

(2.11)

Applying Lemma 1.1 to

$$\left|w_2 - \left(-\frac{1}{(1+2\alpha)}\left(1 - \frac{2\mu(1+3\alpha)}{(1+2\alpha)}\right)B_1 C_0 - \frac{B_2}{B_1}\right)w_1^2\right| \qquad (2.12)$$

yields

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1}{2(1+3\alpha)}\left(1 + \max\left\{1, \left|-\frac{1}{(1+2\alpha)}\left(1 - \frac{2\mu(1+3\alpha)}{(1+2\alpha)}\right)B_1 C_0 - \frac{B_2}{B_1}\right|\right\}\right)$$

(2.13)

Observe that

$$\left|-\frac{1}{(1+2\alpha)}\left(1 - \frac{2\mu(1+3\alpha)}{(1+2\alpha)}\right)B_1 C_0 - \frac{B_2}{B_1}\right| \le B_1 \mid C_0 \mid \left|1 - \frac{2\mu(1+3\alpha)}{(1+2\alpha)}\right| + \left|\frac{B_2}{B_1}\right| \qquad (2.14)$$

and hence we can conclude that

$$\left|a_3-\mu a_2^2\right| \le \frac{B_1}{2(1+3\alpha)}\left(1+\max\left\{1, \frac{B_1}{1+2\alpha}\left|1-\frac{2\mu(1+3\alpha)}{(1+2\alpha)}\right|+\left|\frac{B_2}{B_1}\right|\right\}\right) \qquad (2.15)$$

For $\mu = 0$, the above will reduce to the estimate of $|a_3|$. □

Theorem 2.2. If $f \in A$ satisfies

$$\frac{\alpha z^2 f''(z)+zf'(z)}{f(z)}-1 << \phi(z)-1 \qquad (2.16)$$

then the following inequalities hold:

$$|a_2| \le \frac{B_1}{1+2\alpha}, \quad |a_3| \le \frac{B_1}{2(1+3\alpha)}\left[1+\frac{B_1}{1+2\alpha}+\left|\frac{B_2}{B_1}\right|\right] \qquad (2.17)$$

and for any complex number μ,

$$\left|a_3-\mu a_2^2\right| \le \frac{B_1}{2(1+3\alpha)}\left(1+\frac{B_1}{1+2\alpha}\left|1-\frac{2\mu(1+3\alpha)}{(1+2\alpha)}\right|+\left|\frac{B_2}{B_1}\right|\right) \qquad (2.18)$$

Proof. The result follows by taking $w(z) = z$ in the proof of Theorem 2.1. □

Theorem 2.3. Let $\alpha \ge 0$. If $f \in A$ belongs to $M_q(\alpha, \lambda, \phi)$ then

$$|a_2| \le \frac{B_1}{(1+\alpha)(1+\lambda)},$$

$$|a_3| \le \frac{B_1}{(2+\alpha)(1+2\lambda)}\left(1+\max\left\{1, \frac{B_1}{(1+\alpha)^2(1+\lambda)^2}\left|\frac{(1-\alpha)(2+\alpha)}{2}+\lambda(3+\alpha)\right|+\left|\frac{B_2}{B_1}\right|\right\}\right)$$

and

$$\left|a_3-\mu a_2^2\right| \le \frac{B_1}{(2+\alpha)(1+2\lambda)}\left(1+\max\left\{1, \frac{B_1}{(1+\alpha)^2(1+\lambda)^2}\right.\right.$$

$$\left.\left.\times\left|\frac{(1-\alpha)(2+\alpha)}{2}+\lambda(3+\alpha)-\mu(2+\alpha)(1+2\lambda)\right|+\left|\frac{B_2}{B_1}\right|\right\}\right) \qquad (2.19)$$

Proof. If $f \in M_q(\alpha, \lambda, \phi)$, for $\lambda \ge 0$ then there are analytic functions $\underline{\varphi}$ and w, with $\left|\underline{\varphi}(z)\right| \le 1$, $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha}+\lambda\left[1+\frac{zf''(z)}{f(z)}-\frac{zf'(z)}{f(z)}+\alpha\left(\frac{f'(z)}{f(z)}-1\right)\right]-1=\underline{\varphi}(z)(\phi(w(z))-1) \qquad (2.20)$$

A computation shows that

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha}=1+a_2(1+\alpha)z+(2+\alpha)\frac{z^2}{2}[2a_3+(\alpha-1)a_2^2]$$

$$\lambda\left[1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}+\alpha\left(\frac{f'(z)}{f(z)}-1\right)\right]=\lambda[a_2z+z^2(4a_3-3a_2^2)+\alpha(a_2z+z^2(2a_3-a_2^2)] \qquad (2.21)$$

Hence from (2.18), we have

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha}+\lambda\left[1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}+\alpha\left(\frac{f'(z)}{f(z)}-1\right)\right]-1$$

$$= a_2(1+\alpha)(1+\lambda)z + z^2\left[a_3(2+\alpha)(1+2\lambda\lambda + a_2^2\left(\frac{(\alpha-1)(2+\alpha)}{2} - \lambda(3+\alpha)\right)\right] + \cdots \quad (2.22)$$

It then follows from relation (2.17) and (2.19) that

$$a_2 = \frac{B_1 C_0 w_1}{(1+\alpha)(1+\lambda)},$$

$$a_3 = \frac{1}{(2+\alpha)(1+2\lambda)}\Big\{ B_1 C_1 w_1 + B_1 C_0 w_2$$

$$+ C_0\left[B_2 + \frac{B_2^2 C_0}{(1+\alpha)^2(1+\lambda)^2}\left(\frac{(1-\alpha)(2+\alpha)}{2} + \lambda(\alpha+3)\right) w_1^2\right]\Big\} \quad (2.23)$$

We can then conclude the proof by proceeding similarly as previous theorem. □

Theorem 2.4. If $f \in A$ satisfies

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left[1 + \frac{zf''(z)}{f(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right] - 1 << \phi(z) - 1 \quad (2.24)$$

then the following inequalities hold:

$$|a_2| \le \frac{B_1}{(1+\alpha)(1+\lambda)},$$

$$|a_3| \le \frac{B_1}{(2+\alpha)(1+2\lambda)}\left(1 + \frac{B_1}{(1+\alpha)^2(1+\lambda)^2}\left|\frac{(1-\alpha)(2+\alpha)}{2} + \lambda(\alpha+3)\right| + \left|\frac{B_2}{B_1}\right|\right) \quad (2.25)$$

and for any complex number μ,

$$|a_3 - \mu a_2^2| \le \frac{B_1}{(2+\alpha)(1+2\lambda)}\left(1 + \frac{B_1}{(1+\alpha)^2(1+\lambda)^2}\right.$$

$$\left.\left|\frac{(1-\alpha)(2+\alpha)}{2} + \lambda(3+\alpha) - \mu(2+\alpha)(1+2\lambda)\right| + \left|\frac{B_2}{B_1}\right|\right) \quad (2.26)$$

Proof. The result follows by taking $w(z) = z$ in the proof of Theorem 2.3. □

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