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Notes on TOPSIS Method

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ABSTRACT: *The paper demonstrates the application of TOPSIS method using two selected examples. In the first example, it is shown that the best TOPSIS solution is neither closest to the positive ideal solution nor the farthest from the negative ideal solution. In many works on TOPSIS method stands as follows: "The basic principle is that the chosen alternative should have the shortest distance from the positive ideal solution and the longest distance from the negative ideal solution".*
Keywords: *decision matrix, negative ideal solution, positive ideal solution*

I. INTRODUCTION

Multi-criteria decision making.

In a general sense, it is the aspiration of human being to make "calculated" decision in a position of multiple selection. In scientific terms, it is the intention to develop analytical and numerical methods that take into account multiple alternatives with multiple criteria.

TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) is one of the numerical methods of the multi-criteria decision making. This is a broadly applicable method with a simple mathematical model. Furthermore, relying on computer support, it is very suitable practical method. The method is applied in the last three decades (on the history of TOPSIS see [4], [3]), and there are many papers on its applications (see [11], [8], [9]).

Description of the problem.

Given m options (alternatives) A_i, each of which depends on n parameters (criteria) x_j whose values are expressed with positive real numbers x_{ij}. The best option should be selected.

Mathematical model of the problem.

Initially, the parameter values x_{ij} should be balanced according to the procedure of normalization. Suppose that a_{ij} are the normalized parameter values. Then each option A_i is expressed as the point $A_i(a_{i1},\dots,a_{in}) \in \mathrm{R}^n$. Selecting the most optimal value $a_j^* \in \{a_{1j},\dots,a_{mj}\}$ for every parameter x_j, we determine the positive ideal solution $A^* = (a_1^*,\dots,a_n^*)$. The opposite is the negative ideal solution $A^\lozenge = (a_1^\lozenge,\dots,a_n^\lozenge)$. The positive and negative ideal solution are also denoted by A^+ and A^-. The decision on the order of options is made respecting the order of numbers

$$D_i^* = \frac{d(A_i,A^\lozenge)}{d(A_i,A^*)+d(A_i,A^\lozenge)} = \frac{1}{d(A_i,A^*)/d(A_i,A^\lozenge)+1}. \qquad (1)$$

The option A_{i_1} is the **best solution** if $\max\{D_1^*,D_2^*,\dots,D_m^*\} = D_{i_1}^*$, and the option A_{i_2} is the **worst solution** if $\min\{D_1^*,D_2^*,\dots,D_m^*\} = D_{i_2}^*$. The other options are between these two extremes. The maximum distance $D^* = \max_{i=1,\dots,m} D_i^*$ is usually called TOPSIS metric.

Geometrical image of the problem.

Fig. 1 shows the initial arrangement of alternatives in TOPSIS method for $n = 2$. Parameter $x_1 = x_1^*$ has a monotonically increasing preference, and parameter $x_2 = x_2^\lozenge$ has a monotonically decreasing preference. The positive A^* and negative $A^\lozenge$ ideal solution are located at diagonally opposite positions. The best solution is the alternative A_7.

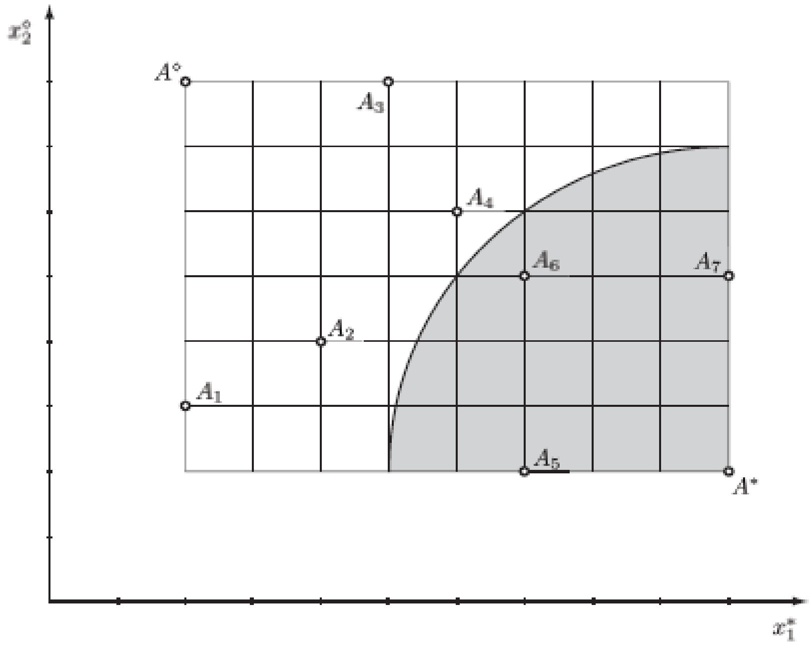

Figure 1. Geometrical representation of TOPSIS method

TOPSIS is a compensatory method. These kinds of methods allow the compromise between different criteria, where a bad result in one criterion can be compensated by a good result in another criterion. An assumption of TOPSIS method is that each criterion has either a monotonically increasing or decreasing preference. Due to the possibility of criteria modelling, compensatory methods, certainly including TOPSIS, are widely used in various sectors of multi-criteria decision making (see [10], [2], [1]).

II. COMPUTATIONAL PROCEDURE FOR TOPSIS METHOD

Problem.

We examine m alternatives $A_1, \dots, A_m$. Each alternative A_i respects n criteria $x_1, \dots, x_n$ which are expressed with positive numbers x_{ij}. The criteria $x_1, \dots, x_k$ are benefit (monotonically increasing preference), and criteria $x_{k+1}, \dots, x_n$ are non-benefit (monotonically decreasing preference). Weights w_j of the criteria x_j are given so that $\sum_{j=1}^{n} w_j = 1$. It is necessary to select the most optimal alternative.

Initial Table and Decision Matrix.

For better visibility, the given alternatives, criteria and its weights are placed in the table (see Table 1).

CRITERIA	x_1 cr. 1	x_2 cr. 2	… …	x_n cr. n
weights	w_1	w_2	…	w_n
A_1	x_{11}	x_{12}	…	x_{1n}
A_2	x_{21}	x_{22}	…	x_{2n}
⋮	⋮	⋮	⋱	⋮
A_m	x_{m1}	x_{m2}	…	x_{mn}

Table 1. Initial table for TOPSIS method

The given numbers x_{ij} and their matrix

$$\mathrm{X} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix} \tag{2}$$

must be balanced, since the numbers x_{ij} present values of different criteria with different measuring units. One must also take into account the given weights w_j of the criteria x_j. First, the measuring numbers x_{ij} of the criteria x_j are replaced with the normalized or relative numbers

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}} \tag{3}$$

belonging to the open interval $\langle 0,1 \rangle$. Then, according to the share $w_j x_j$ of the criteria x_j, the normalized numbers r_{ij} are replaced with the weighted normalized numbers

$$a_{ij} = w_j r_{ij} = w_j \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}} \tag{4}$$

belonging to $\langle 0,1 \rangle$. The further data processing uses the weighted normalized decision matrix

$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}. \tag{5}$$

If all weights w_j are mutually equal, in which case $w_j = 1/n$, the numbers r_{ij} can be applied in the matrix A as the numbers a_{ij}.

Working Table.

The weighted normalized decision matrix A and all the data that will be calculated, we try to write in one table.

CRIT ERIA	x_1^* cr. 1	x_2^* cr. 2	… …	x_k^* cr. k	$x_{k+1}^\diamond$ cr. $k+1$	… …	$x_n^\diamond$ cr. n	d^* dips	$d^\diamond$ dins	D^* topm
A_1	a_{11}	a_{12}	…	a_{1k}	$a_{1\,k+1}$	…	a_{1n}	d_1^*	$d_1^\diamond$	D_1^*
A_2	a_{21}	a_{22}	…	a_{2k}	$a_{2\,k+1}$	…	a_{2n}	d_2^*	$d_2^\diamond$	D_2^*
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	a_{m1}	a_{m2}	…	a_{mk}	$a_{m\,k+1}$	…	a_{mn}	d_m^*	$d_m^\diamond$	D_m^*
A^*	a_1^*	a_2^*	…	a_k^*	a_{k+1}^*	…	a_n^*	A^*	$A^\diamond$	$d^* \sim d^\diamond$
$A^\diamond$	$a_1^\diamond$	$a_2^\diamond$	…	$a_k^\diamond$	$a_{k+1}^\diamond$	…	$a_n^\diamond$			

Table 2. Working table for TOPSIS method

The coordinates a_j^* of the positive ideal solution $A^* = (a_1^* \, a_2^* \ldots a_n^*)$ are chosen using the formula

$$a_j^* = \begin{cases} \max\limits_i a_{ij} & \text{for} \quad j = 1, \ldots, k \\ \min\limits_i a_{ij} & \text{for} \quad j = k+1, \ldots, n. \end{cases} \tag{6}$$

If some alternative A_{i_0} is equal to A^*, then it is obvious that the alternative A_{i_0} is the best solution. If it is not, then we continue the procedure.

The coordinates $a_j^\lozenge$ of the negative ideal solution $A^\lozenge = (a_1^\lozenge \, a_2^\lozenge \ldots a_n^\lozenge)$ are chosen applying the formula

$$a_j^\lozenge = \begin{cases} \min\limits_i a_{ij} & \text{for} \quad j = 1, \ldots, k \\ \max\limits_i a_{ij} & \text{for} \quad j = k+1, \ldots, n. \end{cases} \tag{7}$$

The numbers d_i^* of the column $d^* = (d_1^* \, d_2^* \ldots d_m^*)^{\mathrm{T}}$ are the distances from the points A_i to the point A^*, which is calculated by the formula

$$d_i^* = d(A_i, A^*) = \sqrt{\sum_{j=1}^{n} \left(a_{ij} - a_j^*\right)^2}. \tag{8}$$

The numbers $d_i^\lozenge$ of the column $d^\lozenge = (d_1^\lozenge \, d_2^\lozenge \ldots d_m^\lozenge)^{\mathrm{T}}$ are the distances from the points A_i to the point $A^\lozenge$, which is calculated by the formula

$$d_i^\lozenge = d(A_i, A^\lozenge) = \sqrt{\sum_{j=1}^{n} \left(a_{ij} - a_j^\lozenge\right)^2}. \tag{9}$$

The numbers D_i^* of the column $D^* = (D_1^* \, D_2^* \ldots D_m^*)^{\mathrm{T}}$ are the relative distances of the points A_i respecting the points A^* and $A^\lozenge$, which is expressed by the formula

$$D_i^* = \frac{d_i^\lozenge}{d_i^* + d_i^\lozenge} = \frac{d(A_i, A^\lozenge)}{d(A_i, A^*) + d(A_i, A^\lozenge)}. \tag{10}$$

If $\max\{D_1^*, D_2^*, \ldots, D_m^*\} = D_{i_1}^*$, then we accept the alternative A_{i_1} as the **best solution**. If $\min\{D_1^*, D_2^*, \ldots, D_m^*\} = D_{i_2}^*$, then we accept the alternative A_{i_2} as the **worst solution**.

III. TWO EXAMPLES OF USING TOPSIS METHOD

Example 1. Four alternatives with three criteria are given in Table 3. The criteria x_1 and x_2 are benefit, and the criterion x_3 is non-benefit. The weights of the criteria are equal. Decide which alternative is the best.

CRIT ERIA	x_1 *cr.* 1	x_2 *cr.* 2	x_3 *cr.* 3
weights	1/3	1/3	1/3
A_1	7	60	5
A_2	9	60	6
A_3	5	70	9
A_4	4	80	3

Table 3. Initial table for Example 1

Since the weights are equal, we can use the normalized decision matrix A with the elements

$$a_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{4} x_{ij}^2}} \tag{11}$$

for $i = 1,2,3,4$ and $j = 1,2,3$. In this case, the matrix A reads as follows:

$$\mathrm{A} = \begin{bmatrix} 0,535 & 0,441 & 0,407 \\ 0,688 & 0,441 & 0,488 \\ 0,382 & 0,515 & 0,732 \\ 0,306 & 0,588 & 0,244 \end{bmatrix}. \tag{12}$$

Relying on the matrix A, we have to determine two rows (A^*, $A^\Diamond$) and three columns (d^*, $d^\Diamond$, D^*) in the working table.

CRIT ERIA	x_1^* *cr.* 1	x_2^* *cr.* 2	$x_3^\Diamond$ *cr.* 3	d^* *dips*	$d^\Diamond$ *dins*	D^* *topm*
A_1	0, 535	$0,441^\Diamond$	0, 407	0, 268	0, 398	0, 598
A_2	$0,688^*$	$0,441^\Diamond$	0, 488	0, 285	0, 453	$0,614^*$
A_3	0, 382	0, 515	$0,732^\Diamond$	0, 581	0, 106	$0,154^\Diamond$
A_4	$0,306^\Diamond$	$0,588^*$	$0,244^*$	0, 382	0, 510	0, 572
A^*	$0,688^*$	$0,588^*$	$0,244^*$	A^*	$A^\Diamond$	$d^* \sim d^\Diamond$
$A^\Diamond$	$0,306^\Diamond$	$0,441^\Diamond$	$0,732^\Diamond$			

Table 4. Working table for Example 1

According to the formula in (6), the positive ideal solution $A^* = (a_1^* a_2^* a_3^*)$ contains the greatest numbers of the first and second column of A, and the smallest number of the third column of A.

According to the formula in (7), the negative ideal solution $A^\Diamond = (a_1^\Diamond a_2^\Diamond a_3^\Diamond)$ contains the smallest numbers of the first and second column of A, and the greatest number of the third column of A.

The distances $d^* = (d_1^* d_2^* d_3^* d_4^*)^{\mathrm{T}}$, from the alternatives A_i to the positive ideal solution A^*, are calculated by the formula in (8) with $n = 3$, so

$$d_i^* = d(A_i, A^*) = \sqrt{\sum_{j=1}^{3}\left(a_{ij} - a_j^*\right)^2}. \tag{13}$$

The distances $d^\Diamond = (d_1^\Diamond d_2^\Diamond d_3^\Diamond d_4^\Diamond)^{\mathrm{T}}$, from the alternatives A_i to the negative ideal solution $A^\Diamond$, are calculated by the formula in (9) with $n = 3$, so

$$d_i^\Diamond = d(A_i, A^\Diamond) = \sqrt{\sum_{j=1}^{3}\left(a_{ij} - a_j^\Diamond\right)^2}. \tag{14}$$

The relative distances $D^* = (D_1^* D_2^* D_3^* D_4^*)^{\mathrm{T}}$ of the alternatives A_i respecting the positive ideal solution A^* and negative ideal solution $A^\Diamond$ are calculated using the formula in (10), so

$$D_i^* = \frac{d_i^\Diamond}{d_i^* + d_i^\Diamond}. \tag{15}$$

Applying the last tree columns of the Table 4 we have the following three preferred orders of alternatives:

$A_2 A_1 A_4 A_3$ by the column of D^* from the largest to smallest number

$A_4 A_2 A_1 A_3$ by the column of $d^\Diamond$ from the largest to smallest number

$A_1 A_2 A_4 A_3$ by the column of d^* from the smallest to largest number

TOPSIS method prefers the first order respecting the column of D^*.

In the following example we have the most desirable combination: d^*, $d^\Diamond$ and D^* point to one and the same best solution.

Example 2. 1 Six alternatives with five criteria and their weights are given in Table 5. The criteria x_1, x_2, x_3 are benefit, and the criteria x_4, x_5 are non-benefit. Select the best alternative.

CRIT ERIA	x_1 cr. 1	x_2 cr. 2	x_3 cr. 3	x_4 cr. 4	x_5 cr. 5
weights	0,30	0,10	0,20	0,25	0,15
A_1	6	7	0,4	8	30
A_2	8	5	0,3	7	80
A_3	5	4	0,8	3	70
A_4	4	9	0,7	6	50
A_5	9	4	0,5	7	60
A_6	3	4	0,6	9	20

Table 5. Initial table for Example 2

Weighted normalized decision matrix A with the elements

$$a_{ij} = w_j \frac{x_{ij}}{\sqrt{\sum_{i=1}^{6} x_{ij}^2}} \tag{16}$$

for $i = 1, \ldots, 6$ and $j = 1, \ldots, 5$ stands as follows:

$$A = \begin{bmatrix} 0,118 & 0,049 & 0,057 & 0,118 & 0,033 \\ 0,158 & 0,035 & 0,043 & 0,103 & 0,088 \\ 0,097 & 0,028 & 0,113 & 0,044 & 0,077 \\ 0,079 & 0,063 & 0,099 & 0,088 & 0,055 \\ 0,178 & 0,028 & 0,071 & 0,103 & 0,066 \\ 0,059 & 0,028 & 0,085 & 0,133 & 0,022 \end{bmatrix}. \quad (17)$$

The positive ideal solution $A^* = (a_1^* a_2^* a_3^* a_4^* a_5^*)$ contains the greatest numbers of the first, second and third column of A, and the smallest numbers of the fourth and fifth column of A.

The negative ideal solution $A^\Diamond = (a_1^\Diamond a_2^\Diamond a_3^\Diamond a_4^\Diamond a_5^\Diamond)$ contains the smallest numbers of the first, second and third column of A, and the greatest numbers of the fourth and fifth column of A.

The distances $d^* = (d_1^* d_2^* d_3^* d_4^* d_5^* d_6^*)^T$, from the alternatives A_i to the positive ideal solution A^*, are calculated applying the distance formulas in (8) with $n = 5$.

The distances $d^\Diamond = (d_1^\Diamond d_2^\Diamond d_3^\Diamond d_4^\Diamond d_5^\Diamond d_6^\Diamond)^T$, from the alternatives A_i to the negative ideal solution $A^\Diamond$, are calculated applying the distance formulas in (9) with $n = 5$.

The relative distances $D^* = (D_1^* D_2^* D_3^* D_4^* D_5^* D_6^*)^T$ of the alternatives A_i respecting the positive ideal solution A^* and negative ideal solution $A^\Diamond$ are determined using the quotient formulas in (10).

CRIT ERIA	x_1^* cr. 1	x_2^* cr. 2	x_3^* cr. 3	$x_4^\diamond$ cr. 4	$x_5^\diamond$ cr. 5	d^* dips	$d^\diamond$ dins	D^* topm
A_1	0,118	0,049	0,057	0,118	0,033	0,112	0,086	0,434
A_2	0,158	0,035	0,043$^\diamond$	0,103	0,088$^\diamond$	0,118	0,104	0,468
A_3	0,097	0,028$^\diamond$	0,113*	0,044*	0,077	0,104	0,120	0,536
A_4	0,079	0,063*	0,099	0,088	0,055	0,114	0,088	0,436
A_5	0,178*	0,028$^\diamond$	0,071	0,103	0,066	0,092	0,128	0,582*
A_6	0,059$^\diamond$	0,028$^\diamond$	0,085	0,133$^\diamond$	0,022*	0,155	0,078	0,335$^\diamond$
A^*	0,178*	0,063*	0,113*	0,044*	0,022*	A^*	$A^\diamond$	$d^* \sim d^\diamond$
$A^\diamond$	0,059$^\diamond$	0,028$^\diamond$	0,043$^\diamond$	0,133$^\diamond$	0,088$^\diamond$			

Table 6. Working table for Example 2

Applying the last tree columns of the Table 6 we have the following three preferred orders of alternatives:

$A_5 A_3 A_2 A_4 A_1 A_6$ by the column of D^* from the largest to smallest number

$A_5 A_3 A_2 A_4 A_1 A_6$ by the column of $d^\Diamond$ from the largest to smallest number

$A_5 A_3 A_1 A_4 A_2 A_6$ by the column of d^* from the smallest to largest number

The first order is preferred by TOPSIS method.

IV. CONCLUSION

In the method presenting it is important to find examples that adequately show its meaning and application. After that, the generalizations of the method can be implemented to extend its applications. In the case of TOPSIS method the first generalization refers to the processing of insufficiently precise data, namely, fuzzy data (see [6], [5], [1]).

The second generalization applies to the norm and metric. Let $p \geq 1$ be a real number. Using the p -norm in the normalization procedure, we get

$$r_{ij} = \frac{x_{ij}}{\|(x_{1j},\ldots,x_{mj})\|_p} = \frac{x_{ij}}{\sqrt[p]{\sum_{i=1}^{m} |x_{ij}|^p}}. \quad (18)$$

Then, using the p -metric in the distance calculation, we have

$$d_i^* = d_p(A_i, A^*) = \sqrt[p]{\sum_{j=1}^{n} |a_{ij} - a_j^*|^p}. \quad (19)$$

The max-norm and max-metric can also be applied in the computational procedure of TOPSIS method .

REFERENCES

[1] A. Balin, P. Alcan, and H. Basligil, Co performance comparison on CCHP systems using different fuzzy multi criteria decision making models for energy sources, Fuelling the Future, pp. 591-595, 2012.

[2] I. B. Huang, J. Keisler, and I. Linkov, Multi-criteria decision analysis in environmental science: ten years of applications and trends, Science of the Total Environment 409, pp. 3578-3594, 2011.

[3] C. L. Hwang, Y. J. Lai, and T. Y. Liu, A new approach for multiple objective decision making, Computers and Operational Research 20, pp. 889-899, 1983.

[4] C. L. Hwang, and K. Yoon, Multiple Attribute Decision Making: Methods and Applications, Berlin Heidelberg New York, Springer-Verlag, 1981.

[5] G. R. Jahanshahloo, F. Hosseinzadeh Lotfi, and M. Izadikhah, Extension of the TOPSIS method for decision-making problems with fuzzy data, Applied Mathematics and Computation, pp. 1544-1551, 2006.

[6] Y. J. Lai, T. Y. Liu, and C. L. Hwang, Fuzzy Mathematical Programming: Methods and Applications, Berlin Heidelberg New York, Springer-Verlag, 1992.

[7] Y. J. Lai, T. Y. Liu, andC. L. Hwang, TOPSIS for MODM, European Journal of Operational Research 76, pp. 486-500, 1994.

[8] G. H. Tzeng, and J. J. Huang, Multiple Attribute Decision Making: Methods and Applications, New York, CRC Press, 2011.

[9] J. Xu, and Z. Tao, Rough Multiple Objective Decision Making, New York, CRC Press, 2012.

[10] K. A. Yoon, A reconciliation among discrete compromise situations, Journal of Operational Research Society 38, pp. 277-286, 1987.

[11] K. P. Yoon, and C. Hwang, Multiple Attribute Decision Making: An Introduction, California, SAGE Publications, 1995.

[12] E. K. Zavadskas, A. Zakarevicius, and J. Antucheviciene, Evaluation of ranking accuracy in multi-criteria decisions, Informatica 17, pp. 601-618, 2006.