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Numerical Minimal Distance between Two Arbitrary Catenaries in 3D Space

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Abstract
Catenary fixed by two points is a curve which looks like a parabola but actually it is not. In this paper, we are interested in specifying the minimal distance between two catenaries in the 3D space via numerical optimization. Such a minimal distance plays an important role in many real-world applications, such as the high-voltage power transmission engineering. We first derive the catenary function for arbitrary hanging points in the 3D space based on the idea of balance of the tension force. Then, we give the mathematical formula for computing the minimal distance, which concerns a minimization problem and can be solved efficiently by existing optimization solvers. Numerical results are given to illustrate that the mathematical formula is flexible tohandle two representative situations.

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I. Introduction

A catenary is the shape that a rope or chain will naturally converge to, when suspended at its two ends. It is not a coincidence that the name catenary itself comes from the Latin catenariawhich indeed means chain. The following is a brief history of catenary. In the late 17th century, Hooke identified the shape of catenary with that of an inverted arch. Not long after, in response to a challenge proposed by Jacob Bernoulli, the solution of the catenary was found independently by Johann Bernoulli, Huygens, and Leibniz [2]. In the mid-19th century, the catenaries were recognized to be solutions of a broader dynamical problem: determining the shape equilibria of a moving string subjected to a uniform body force. This was perhaps first recognized by a Tripos examiner in 1854 and Routh incorporated the fact in a mechanics text [9]. By that time, the discovery had also been made by Airy (Astronomer Royal) and Thomson (Lord Kelvin), whose impetus to study this problem was the massive failure of the first attempt to lay transatlantic telegraph cable [1, 10].

Nowdays, applications of catenary spans widely. For example, after the events occurring at the Ronan Point apartment building in London, the Murrah Federal Building in Oklahoma

City, and the World Trade Center in New York City, progressive collapse (the extensive or complete collapse of a structure) resulting from the failure of one or a small number of structural components has become a focus of research efforts and design considerations. Retrofitting cable is one method to enhance existing frames or replace the post-mechanism beam load resistance. The cables are located linearly along the beam geometry and are affixed at beam supports and the crucial mathematical tool for modeling the shape of the cables, which is catenary [5]. There is also application of catenary in thermo-mechanical industry. Experts in this industry want to improve the energy efficiency of compressors. Thus, they are objective to propose an alteration in the geometry of a hermetic reciprocating compressor using a catenary curve, which will be applied in all the surfaces of the compressor cylinder block and will reduce the consumption of energy [3]. Other interesting research about catenary can be found in [4, 8].

Here, we are interested the high-voltage power transmission engineering, for which the shape of the transmission line between two pylons is a typical catenary. The shape is determined by the hanging locations and the length of the cable. There are two important quantities for the catenary in this filed. First, the minimal distance between a single cable and the ground. Second, the minimal distance between two arbitrary cables in 3D space. These two minimal distances are the crucial issues for safety consideration in practice. The first distance was already extensively studied and mathematical formulas and softwares are available in literature. But the research for the second minimal distance is rare. In this paper, by using the optimization tool in Matlab, we determine the minimal distance between arbitrary catenaries in the 3D space (i.e., the real-world situation). The basic method is to model the catenary in a plane (i.e., in the 2D space) and then transform it to a 3D space. The minimal distance is then computed by the fmincon command in Matlab with the distance function between any two points located on the catenaries.

The rest of the paper is organized as follows. In Section 2, we present the details for constructing the function for the catenary via the idea of force balance. In Section 3, we present the mathematical formula for computing the minimal distance between two arbitrary catenaries in the 3D space. Some numerical results are also given in this section. Finally,we conclude this paper in Section 4.

II. The catenary function

In this section, we present the mathematics for the catenary. There are many different approaches for deriving the function of a catenary and here we introduce a simple one based on *force balance*. The derivation of the catenary function needs the following mean-value theorem.

Lemma 1 ([7]) *Let $f(z)$ be a differentiable function in the interval $z \in (a, b)$. Then, for any $z \in (a, b)$ and any $z_0 \in (a, b)$ sufficiently close to z, it holds that*

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o((z - z_0)^2),$$

i.e., $f(z) \approx f(z_0) + f'(z_0)(z - z_0)$.

Based on Lemma 1, for $c \approx 0$ we have

$$\sqrt{1+c} = 1 + \left(\frac{1}{2\sqrt{1+c}}\Big|_{c=0} \right) c + o(c^2) = 1 + \frac{c}{2} + o(c^2),$$

i.e., $\sqrt{1+c} \approx 1 + \frac{c}{2}$. Such a simple approximation will be used frequently in the following.

2.1 Force balance analysis for the catenary in 2D

In this subsection, we try to establish the catenary function in 2D (i.e., in a plane), by using the balance relationship for the force at an arbitrary point along the catenary. Then, in the next subsection we describe the catenary function in a real 3D space. For convenience, we suppose the left hanging point of the catenary is fixed on the y-axis as shown in Figure 2.1. In this figure, we pick a point x, at which we suppose the tension force is $T(x)$ and the height of the catenary is $y(x)$. The direction of the tension force is denoted by an angle θ_1 with the horizontal line. Similarly, with a small increment dx for x, we suppose the tension force and the height of the catenary at $x + dx$ is $T(x + dx)$ and $y(x + dx)$, respectively. The direction of the tension force at $x + dx$ is denoted by an angle θ_2 with the horizontal line. In the following, we denote by ρ the mass per unit length of the catenary and by g the acceleration of gravity.

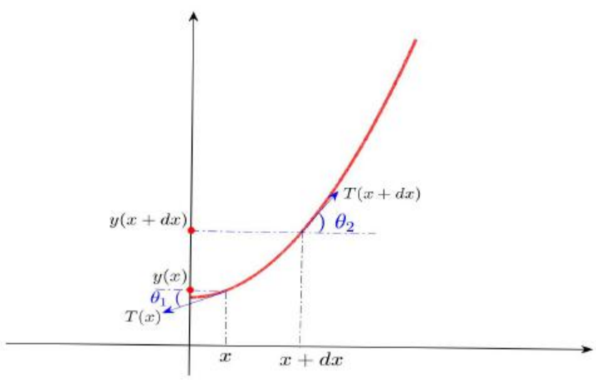

Figure 2.1: The illustration of the tension force and the coordinates of the catenary.

We now explore the relationship between the tension force $T(x)$ and the height $y(x)$, which consists of the following 3 steps.

• **Step-I**. The balance of the horizontal and vertical forces at x and $x+dx$ leads to the following two equations

$$\begin{aligned} T(x+dx)\cos(\theta_2) &= T(x)\cos(\theta_1), \\ T(x+dx)\sin(\theta_2) &= T(x)\sin(\theta_1) + \frac{g\rho dx}{\cos(\theta_1)}. \end{aligned} \tag{2.1}$$

Squaring and adding the two equations in (2.1) gives

$$T^2(x+dx) \approx T^2(x) + 2T(x)g\rho\tan(\theta_1)dx.$$

By using Lemma 1 it holds $T(x+dx) \approx T(x) + T'(x)dx$. Therefore,

$$T^2(x+dx) \approx T^2(x) + 2T(x)T'(x)dx + (T'(x))^2dx^2 \approx T^2(x) + 2T(x)T'(x)dx.$$

This implies

$$T^2(x) + 2T(x)T'(x)dx \approx T^2(x) + 2T(x)g\rho\tan(\theta_1)dx \Rightarrow T'(x) \approx g\rho\tan(\theta_1) = g\rho y'(x).$$

To get the mathematical model of the catenary, we simply regard $T'(x) = g\rho y'(x)$, which after integration leads to

$$T(x) = g\rho y(x) + c_1, \tag{2.2}$$

where c_1 is free constant. This is the first important relationship between $T(x)$ and $y(x)$.

• **Step-II**. Starting from the point $(x, y(x))$ on the catenary, a small increase of x leads to a linear increase of $y(x)$ with horizontal angle θ_1. Hence, it holds

$$y'(x) = \frac{dy}{dx} = \tan(\theta_1). \tag{2.3a}$$

Similarly, starting from the point $(x+dx, y(x+dx))$ on the catenary, a small increase of the $x+dx$ leads to a linear increase of $y(x+dx)$ with horizontal angle θ_2. Therefore

$$y'(x+dx) = \tan(\theta_2). \tag{2.3b}$$

Becase

$$\tan(\theta_1) = \frac{\sin(\theta_1)}{\cos(\theta_1)}, \quad \tan(\theta_2) = \frac{\sin(\theta_2)}{\cos(\theta_2)},$$

we have

$$1 + \tan^2(\theta_1) = 1 + \frac{\sin^2(\theta_1)}{\cos^2(\theta_1)} = \frac{1}{\cos^2(\theta_1)}, \quad 1 + \tan^2(\theta_2) = 1 + \frac{\sin^2(\theta_2)}{\cos^2(\theta_2)} = \frac{1}{\cos^2(\theta_2)}.$$

This, together with (2.3a) and (2.3b), gives

$$\cos(\theta_1) = \frac{1}{\sqrt{1+\tan^2(\theta_1)}} = \frac{1}{\sqrt{1+(y'(x))^2}}, \quad \cos(\theta_2) = \frac{1}{\sqrt{1+\tan^2(\theta_2)}} = \frac{1}{\sqrt{1+(y'(x+dx))^2}}.$$

Now, substituting $\cos(\theta_1)$ and $\cos(\theta_2)$ into the first equation in (2.1), we have

$$\frac{T(x)}{\sqrt{1+(y'(x))^2}} = \frac{T(x+dx)}{\sqrt{1+(y'(x+dx))^2}}. \tag{2.4}$$

By Lemma 1 (i.e., the mean-value theorem), it holds

$$y'(x+dx) \approx y'(x) + dxy''(x), \ T(x+dx) \approx T(x) + dxT'(x).$$

In (2.4), replacing $y'(x+dx)$ and $T(x+dx)$ by these two approximations leads to

$$\frac{T(x)}{\sqrt{1+(y'(x))^2}} = \frac{T(x)+dxT'(x)}{\sqrt{1+(y'(x)+dxy''(x))^2}}. \tag{2.5}$$

By using Lemma 1 again, the following approximation holds if z_0 is close to z

$$\frac{1}{\sqrt{z}} \approx \frac{1}{\sqrt{z_0}} + \left(-\frac{1}{2z_0^{\frac{3}{2}}}\right)(z - z_0) = \frac{1}{\sqrt{z_0}} - \frac{z - z_0}{2z_0^{\frac{3}{2}}} = \frac{1}{\sqrt{z_0}}\left(1 - \frac{z - z_0}{2z_0}\right). \tag{2.6}$$

In (2.5), because dx is a small quantity it holds

$$\begin{aligned} \frac{1}{\sqrt{1 + [y'(x) + dxy''(x)]^2}} &= \frac{1}{\sqrt{1 + [y'(x)]^2 + 2dxy'(x)y''(x) + dx^2[y''(x)]^2}} \\ &\approx \frac{1}{\sqrt{1 + [y'(x)]^2 + 2dxy'(x)y''(x)}}. \end{aligned} \tag{2.7}$$

Applying (2.6) to (2.7) with $z = 1 + [y'(x)]^2 + 2dxy'(x)y''(x)$ and $z_0 = 1 + [y'(x)]^2$ leads to

$$\begin{aligned} \frac{1}{\sqrt{1 + [y'(x)]^2 + 2dxy'(x)y''(x)}} &\approx \frac{1}{\sqrt{1 + [y'(x)]^2}}\left(1 - \frac{z - z_0}{2z_0}\right) \\ &= \frac{1}{\sqrt{1 + [y'(x)]^2}}\left(1 - \frac{dxy'(x)y''(x)}{1 + [y'(x)]^2}\right). \end{aligned}$$

Substituting this into (2.7) gives

$$\frac{1}{\sqrt{1 + [y'(x) + dxy''(x)]^2}} \approx \frac{1}{\sqrt{1 + [y'(x)]^2}}\left(1 - \frac{dxy'(x)y''(x)}{1 + [y'(x)]^2}\right).$$

This together with (2.5) leads to

$$\begin{aligned} &\frac{T(x)}{\sqrt{1 + (y'(x))^2}} \approx \frac{T(x) + dxT'(x)}{\sqrt{1 + (y'(x))^2}}\left(1 - \frac{dxy'(x)y''(x)}{1 + (y'(x))^2}\right) \\ &= \frac{T(x) + dxT'(x)}{\sqrt{1 + (y'(x))^2}} - \frac{T(x)[dxy'(x)y''(x)]}{[1 + (y'(x))^2]\sqrt{1 + (y'(x))^2}} - dx^2\frac{T'(x)[y'(x)y''(x)]}{[1 + (y'(x))^2]\sqrt{1 + (y'(x))^2}} \\ &\approx \frac{T(x)}{\sqrt{1 + (y'(x))^2}} + dx\frac{T'(x)}{\sqrt{1 + (y'(x))^2}} - dx\frac{T(x)[y'(x)y''(x)]}{[1 + (y'(x))^2]\sqrt{1 + (y'(x))^2}}, \end{aligned}$$

where we have dropped the dx^2-term. We therefore obtain the following relationship

$$T'(x) = \frac{T(x)[y'(x)y''(x)]}{[1 + (y'(x))^2]} \Rightarrow \frac{T'(x)}{T(x)} = \frac{y'(x)y''(x)}{1 + (y'(x))^2}.$$

Now, by noticing

$$(\log T(x))' = \frac{T'(x)}{T(x)}, \quad \frac{y'(x)y''(x)}{1 + (y'(x))^2} = \frac{1}{2}(\log[1 + (y'(x))^2])',$$

we have

$$(\log T(x))' = \frac{1}{2}(\log[1 + (y'(x))^2])'. \tag{2.8}$$

Integrating both sides of (2.8) gives $\log T(x) + c_2 = \frac{1}{2}\log[1 + (y'(x))^2]$, i.e.,

$$2\log T(x) + 2c_2 = \log[1 + (y'(x))^2]. \tag{2.9}$$

Exponentiating both sides of (2.9) leads to

$$e^{2\log T(x) + 2c_2} = e^{\log[1 + (y'(x))^2]} \Rightarrow c_3^2 T^2(x) = 1 + (y'(x))^2, \tag{2.10}$$

where $c_3 = e^{c_2}$.

Now, the two relationship (2.2) and (2.10) gives the follow equations for $T(x)$ and $y(x)$

$$\begin{cases} T(x) = g\rho y(x) + c_1, \\ c_3^2 T^2(x) = 1 + (y'(x))^2. \end{cases} \tag{2.11}$$

In (2.11), substituting the first equation into the second one leads to

$$c_3^2[g\rho y(x) + c_1]^2 = 1 + (y'(x))^2,$$

i.e.,

$$1 + (y'(x))^2 = \alpha^2[y(x) + h]^2, \tag{2.12}$$

where $\alpha = c_3 g\rho$ and $h = \frac{c_1}{g\rho}$. This is a differential equation about $y(x)$ and its solution $y(x)$ is classically known and is given by

$$y(x) = \frac{1}{\alpha}\cosh[\alpha(x+b)] - h, \tag{2.13}$$

where b is a free constant and $\cosh(\cdot)$ is the hyperbolic cosine function defined by

$$\cosh(z) = \frac{e^z + e^{-z}}{2}.$$

Step-III. We next fix the three constants α, b and h by length of the catenary (denoted by l) and the positions of the two hanging points:

$$p_1 : (x_1, y_1), \; p_2 : (x_2, y_2).$$

By the function $y(x)$ of the catenary, we have the following two relationships

$$y_1 = \frac{1}{\alpha}\cosh[\alpha(x_1+b)] - h, \quad y_2 = \frac{1}{\alpha}\cosh[\alpha(x_2+b)] - h. \tag{2.14}$$

The third relationship is about the length of the catenary:

$$\begin{aligned} l &= \int_{x_1}^{x_2} \sqrt{1+(y'(x))^2}dx = \int_{x_1}^{x_2} \sqrt{\alpha^2[y(x)+h]^2}dx = \int_{x_1}^{x_2} \cosh[\alpha(x+b)]dx \\ &= \frac{1}{2\alpha}\left(e^{\alpha(x_2+b)} - e^{\alpha(x_1+b)} - e^{-\alpha(x_2+b)} + e^{-\alpha(x_1+b)}\right). \end{aligned} \tag{2.15}$$

So, from (2.14) and (2.15) we can determine the three parameters α, b and h by the following three nonlinear equations

$$\begin{aligned} l &= \frac{1}{2\alpha}\left(e^{\alpha(x_2+b)} - e^{\alpha(x_1+b)} - e^{-\alpha(x_2+b)} + e^{-\alpha(x_1+b)}\right), \\ y_1 &= \frac{1}{\alpha}\cosh[\alpha(x_1+b)] - h, \\ y_2 &= \frac{1}{\alpha}\cosh[\alpha(x_2+b)] - h. \end{aligned} \tag{2.16}$$

There is no exact mathematical formula for the solution of this nonlinear problems. Alternatively, we can fix the solutions via numerical computation for given parameters (x_1, y_1, x_2, y_2, l).

2.2 Catenary function in real 3D space

Now, suppose we have two points in 3D

$$p : (x_1, y_1, z_1), \; q : (x_2, y_2, z_2),$$

and we need to establish the catenary function in 3D specified by these two points and its length l. The main idea for this goal is to consider the plane specified by the catenary, for which we use a cartesian coordinate with one hanging point located on the vertical axis (see Figure 2.2 for illustration). In such a coordinate system the range of the x-coordinate and y-coordinate of the catenary is

$$x \in [0, \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}], \; y \in [z_1, z_2].$$

So, we can fix the function for the catenary in the cartesian coordinate by the following two points

$$\tilde{p} : (0, z_1), \; \tilde{q} : (\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}, z_2),$$

and the nonlinear equations (2.16) (to get the three parameters α, b and h).

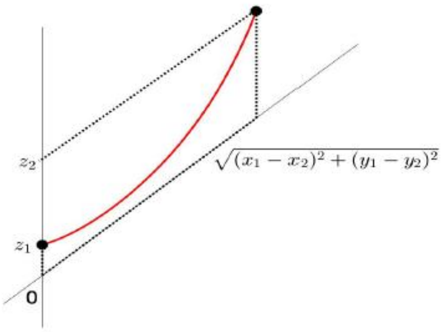

Figure 2.2: The cartesian coordinate for the catenary plane.

Now, we return to the original 3D space. The line fixed by p and q points in the x-y plane is

$$y = \frac{y_1 - y_2}{x_1 - x_2}x + \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2}.$$

For any point along the catenary in the 3D space, denoted by $s : (x, y, z)$, the corresponding point $\tilde{s}$ in the catenary plane (cf. Figure 2.2) is

$$\tilde{s} : (\sqrt{(x - x_1)^2 + (y - y_1)^2}, z)$$

So, in the 3D space the catenary function is

$$\begin{cases} z = \frac{1}{\alpha}\cosh[\alpha(\sqrt{(x - x_1)^2 + (y - y_1)^2} + b)] - h, \\ y = \frac{y_1 - y_2}{x_1 - x_2}x + \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2}, \\ x \in [x_1, x_2]. \end{cases} \tag{2.17}$$

2.3 The illustration codes

The following is an example code for generating the 3D catenary by the Matlab software [6].

```
p=[1/6,1/2,1];  % the first hanging point
q=[2,2,3.5];    % the second hanging point
L=sqrt(sum((p-q)^2))+0.7;  % the length of the catenary
X0=random('unif',0,2,3,1);
ahb=fsolve(@(X) F(X,p,q,L),X0,myopt);  % fix parameters α, b and h
a=abh(1); b=abh(2); h=abh(3);
x1=p(1); y1=p(2); x2=q(1); y2=q(2);
x=x1:0.001:x2;  % discrete x-coordinate of projected catneary
y=((y1-y2)/(x1-x2))*x+(x1*y2-x2*y1)/(x1-x2);
% y-coordinate of projection of the catenary on x-y plane
z=(1/a)*cosh(a*(sqrt((x-x1)^2+(y-y1)^2)+b))-h;  % z-coordinate of catenary
```

The function `F(X, p, q, L)` describes the nonlinear system (2.16)

```
function fun=F(X,p,q,L)
a=X(1);  b=X(2); h=X(3);
x1=p(1); y1=p(2); z1=p(3); x2=q(1); y2=q(2); z2=q(3);
X1=0; X2=sqrt((x1-x2)^2+(y1-y2)^2);  % start and end points of the projected catenary
fun=[L-(0.5/a)*(exp(a*(X2+b))-exp(a*(X1+b))-exp(-a*(X2+b))+exp(-a*(X1+b)));
      z1-((1/a)*cosh(a*(X1+b))-h);
      z2-((1/a)*cosh(a*(X2+b))-h)];  % the three nonlinear equations in (2.16)
end
```

3 Numerical minimal distance between two catenaries

We now consider the minimal distance between two catenaries in the 3D space. As we introduced in Section 1, such a minimal distance plays an important role in many engineering and industry fields, such as the high-voltage power transmission engineering. We suppose the two catenaries are specified by the following data

$$\begin{cases} p:(x_1,y_1,z_1),\ q:(x_2,y_2,z_2),\ \text{length } = l_1, & \text{(first catenary)} \\ \tilde{p}:(\tilde{x}_1,\tilde{y}_1,\tilde{z}_1),\ \tilde{q}:(\tilde{x}_2,\tilde{y}_2,\tilde{z}_2),\ \text{length } = l_2, & \text{(second catenary)} \end{cases} \tag{3.1}$$

and the parameters specified by (2.16) are denoted by (α, b, h) and $(\tilde{\alpha}, \tilde{b}, \tilde{h})$, respectively. Then, according to (2.17) we denote the functions of these two catenaries by

$$\begin{aligned} &\text{first catenary}: \begin{cases} x \in [x_1, x_2], \\ y(x) = \frac{y_1-y_2}{x_1-x_2}x + \frac{x_1y_2-x_2y_1}{x_1-x_2}, \\ z(x) = \frac{1}{\alpha}\cosh[\alpha(\sqrt{(x-x_1)^2+(y(x)-y_1)^2}+b)] - h, \end{cases} \\ &\text{second catenary}: \begin{cases} \tilde{x} \in [\tilde{x}_1, \tilde{x}_2], \\ \tilde{y}(\tilde{x}) = \frac{\tilde{y}_1-\tilde{y}_2}{\tilde{x}_1-\tilde{x}_2}\tilde{x} + \frac{\tilde{x}_1\tilde{y}_2-\tilde{x}_2\tilde{y}_1}{\tilde{x}_1-\tilde{x}_2}, \\ \tilde{z}(\tilde{x}) = \frac{1}{\tilde{\alpha}}\cosh[\tilde{\alpha}(\sqrt{(\tilde{x}-\tilde{x}_1)^2+(\tilde{y}(\tilde{x})-\tilde{y}_1)^2}+\tilde{b})] - \tilde{h}. \end{cases} \end{aligned} \tag{3.2}$$

Let $s:(x,y,z)$ and $\tilde{s}:(\tilde{x},\tilde{y},\tilde{z})$ be any two points located at the first and second catenaries. The distance between s and $\tilde{s}$ is

$$L(s,\tilde{s}) = \sqrt{(x-\tilde{x})^2+(y-\tilde{y})^2+(z-\tilde{z})^2}.$$

From (3.2) we see that (y,z) are functions of x and $(\tilde{y},\tilde{z})$ are functions of $\tilde{x}$, so we can equivalently rewrite the distance L by

$$L(x,\tilde{x}) = \sqrt{(x-\tilde{x})^2+[y(x)-\tilde{y}(\tilde{x})]^2+[z(x)-\tilde{z}(\tilde{x})]^2}. \tag{3.3}$$

Now, by fixing x and varying $\tilde{x}$ from $\tilde{x}_1$ to $\tilde{x}_2$, the minimum of L as a function of $\tilde{x}$ is

$$L_{\min}(x) = \min_{\tilde{x}\in[\tilde{x}_1,\tilde{x}_2]} L(x,\tilde{x}) \left(= \min_{\tilde{x}\in[\tilde{x}_1,\tilde{x}_2]} \sqrt{(x-\tilde{x})^2+[y(x)-\tilde{y}(\tilde{x})]^2+[z(x)-\tilde{z}(\tilde{x})]^2} \right).$$

Next, we get the minimum of $L_{\min}(x)$ for $x \in [x_1, x_2]$, which corresponds to the minimal distance between the two catenaries in 3D space. That is

$$L_{\min}^{1,2} = \min_{x\in[x_1,x_2]} L_{\min}(x) = \sqrt{\min_{x\in[x_1,x_2],\ \tilde{x}\in[\tilde{x}_1,\tilde{x}_2]} (x-\tilde{x})^2+[y(x)-\tilde{y}(\tilde{x})]^2+[z(x)-\tilde{z}(\tilde{x})]^2}.$$

In summary, by letting

$$\text{dis}(x,\tilde{x}) = (x-\tilde{x})^2+[y(x)-\tilde{y}(\tilde{x})]^2+[z(x)-\tilde{z}(\tilde{x})]^2,$$

the minimal distance between the two catenaries is

$$L_{\min}^{1,2} = \sqrt{\min_{x\in[x_1,x_2],\ \tilde{x}\in[\tilde{x}_1,\tilde{x}_2]} \text{dis}(x,\tilde{x})}. \tag{3.4}$$

The minimization problem in (3.4) can be solved very efficiently by many existing solvers, such as the `fmincon` command in Matlab [6].

Example. To finish this section, we show some numerical results for computing the minimal distance by solving the min-problem (3.4) via the command `fmincon` in Matlab [6]. We test two cases using the data given in Table 3.1, where (p_1, q_1) and (p_2, q_2) are the hanging points for the first and second catenaries and l_1, l_2 are the corresponding lengths of the two catenaries. For these two groups of data, the catenaries and the minimal distance are plotted in Figure 3.1 on the left and right, respectively. The locations for the minimal distance are denoted by the star '$*$' markers. We see that the mathematical formula (3.4) can handle two representative situations: the projected lines of the two catenaries on the (x,y)-plane has no interjection (Figure 3.1 on the left) and the projected lines has one interjection point as shown in Figure 3.1 on the right.

Table 3.1: The data used for Figure 3.1 on the left and right

	p_1	q_1	l_1	p_2	q_2	l_2
left subfigure	(1, 0, 3)	(2, 3, 3.5)	4.7045	(0, 1, 3)	(1, 2, 4)	2.2
right subfigure	(1, 0, 2)	(2, 3, 3)	4.8166	(0.5, 2.5, 3)	(2, 0.5, 4)	3.2

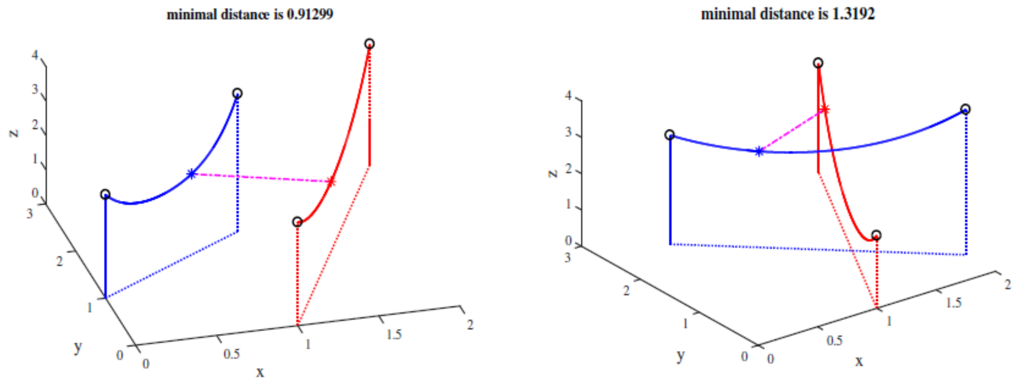

Figure 3.1: The minimal distance between two catenaries in two cases: the catenaries projected onto the (x, y)-plane have interjection point (left subfigure) and have no interjection point (right subfigure). The data for the these two cases are listed in Table 3.1.

IV. Conclusion

In this paper, we revisited the mathematical model for the catenary, which was extensively studied by many authors. The goal of this paper is to compute the minimal distance between two catenaries in the real 3D space. To this end, we first established the catenary function based on the balance of the tension force. For given data (the hanging points and the length of the catenary), such a force balance results in three nonlinear equations (cf. (2.16)) concerning three parameters α, b and h, which completely specify the catenary. Based on the catenary function, we then get the mathematical formula for computing the minimal distance in Section 3. This involves a minimization problem (cf. (3.4)) and can be solved efficiently by many existing optimization solvers, such as fmincon in Matlab.

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