International Journal of Research in Engineering and Science (IJRES)
ISSN (Online): 2320-9364, ISSN (Print): 2320-9356
www.ijres.org Volume 9 Issue 6 ǁ 2021 ǁ PP. 74-78

On The Ternary Quadratic Diophantine Equation

$$x^2 + 3y^2 = 13z^2$$

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ABSTRACT:
The homogeneous ternary quadratic Diophantine equation represented by $x^2 + 3y^2 = 13z^2$ *is studied for finding its non-zero distinct integer solutions. The formulae for generating sequence of integer solutions based on the given solution are exhibited.*
KEYWORDS: *Homogeneous Ternary Quadratic, Integral solutions*

Date of Submission: 25-05-2021 Date of acceptance: 07-06-2021

I. INTRODUCTION

Ternary quadratic equations are rich in variety [1-4, 17-19]. For an extensive review of sizable literature and various problems, one may refer [5-16]. In this communication, we consider yet another interesting homogeneous ternary quadratic equation $x^2 + 3y^2 = 13z^2$ and obtain infinitely many non-trivial integral solutions. Also, the formulae for generating sequence of integer solutions based on the given solution are exhibited.

II. METHOD OF ANALYSIS:

Let x, y, z be any three non-zero distinct integers such that

$$x^2 + 3y^2 = 13z^2 \tag{1}$$

Introducing the linear transformations

$$z = X + T \ , \ x = X + 13T \tag{2}$$

in (1), it leads to

$$y^2 + 52T^2 = 4X^2 \tag{3}$$

We present below different methods of solving (3) and thus, obtain different patterns of integral solutions to (1).

Method: 1

Observe that (3) is satisfied by

$$T = 4rs, \ y = 52r^2 - 4s^2 \ , X = 26r^2 + 2s^2$$

In view of (2), the corresponding values of x and z satisfying (1) are given by

$$x = 26r^2 + 2s^2 + 52rs$$

$$z = 26r^2 + 2s^2 + 4rs$$

Method: 2

Write (3) as the system of double equations as shown in Table: 1 below:

Table: 1 System of double equations

System	1	2	3	4	5	6	7
$2X+y$	T^2	$2T^2$	$13T^2$	$26T^2$	$52T$	$26T$	$13T$
$2X-y$	52	26	4	2	T	2T	4T

Solving each of the system of equations in Table: 1, the corresponding values of X, *y* and T are obtained. Substituting the values of X and T in (2), the respective values of *x* and *z* are determined. For simplicity and brevity, the integer solutions to (1) obtained through solving each of the above system of equations are exhibited.

System :1

$x = k^2 + 26k + 13$

$y = 2k^2 - 26$

$z = k^2 + 2k + 13$

System:2

$x = 2k^2 + 28k + 20$

$y = 4k^2 + 4k + 14$

$z = 2k^2 + 4k + 8$

System:3

$x = 13k^2 + 26k + 1$

$y = 26k^2 - 2$

$z = 13k^2 + 2k + 1$

System :4

$x = 26k^2 + 52k + 20$

$y = 52k^2 + 52k + 12$

$z = 26k^2 + 28k + 8$

System:5

$x = 154k$

$y = 102k$

$z = 57k$

System:6

$x = 20k$

$y = 12k$

$z = 8k$

System :7

$x = 69k$

$y = 18k$

$z = 21k$

Method: 3

(1) is written as

$$3y^2 = 13z^2 - x^2 \quad (4)$$

Assume

$$y = 13a^2 - b^2 \quad (5)$$

Also, 3 is written as

$$3 = \left(2\sqrt{13}+7\right)\left(2\sqrt{13}-7\right) \quad (6)$$

Substituting (5) and (6) in (4) and employing the factorization method, define

$$\sqrt{13}z + x = \left(2\sqrt{13}+7\right)(\sqrt{13}a + b)^2$$

On equating the rational and irrational parts, we have

$$x = 7(13a^2 + b^2) + 52ab \quad , \qquad z = 2(13a^2 + b^2) + 14ab \quad (7)$$

Thus (5) and (7) represent the non-zero distinct integer solutions to (1).

Note: 1

It is worth mentioning here that, in addition to (6), 3 may be represented as below:

(i) $$3 = \frac{\left(\sqrt{13}+1\right)\left(\sqrt{13}-1\right)}{4}$$

(ii) $$3 = \frac{\left(2\sqrt{13}+5\right)\left(2\sqrt{13}-5\right)}{9}$$

Following the procedure presented as above, two more sets of integer solutions to (1) are obtained.

Method: 4

One may write (1) as

$$13z^2 - 3y^2 = x^2 * 1 \quad (8)$$

Assume

$$x = 13a^2 - 3b^2 \quad (9)$$

Write 1 as

$$1 = (\sqrt{13} + 2\sqrt{3})(\sqrt{13} - 2\sqrt{3}) \quad (10)$$

Substituting (10), (9) in (8) and employing the factorization method, define

$$\sqrt{13}z+\sqrt{3}y=\left(\sqrt{13}+2\sqrt{3}\right)\left(\sqrt{13}a+\sqrt{3}b\right)^2$$

On equating the rational and irrational parts, we have

$$z=\left(13a^2+3b^2+12ab\right)\ ,\ y=2(13a^2+3b^2)+26ab \qquad (11)$$

Thus, (9) and (11) represent the non-zero distinct integer solutions to (1).

Note: 2

It is worth mentioning here that, in addition to (10), 1 may be represented as below:

(i) $$1=\frac{\left(2\sqrt{13}+3\sqrt{3}\right)\left(2\sqrt{13}-3\sqrt{3}\right)}{25}$$

(ii) $$1=\frac{\left(2\sqrt{13}+\sqrt{3}\right)\left(2\sqrt{13}-\sqrt{3}\right)}{49}$$

Following the procedure presented as above, two more sets of integer solutions to (1) are obtained.

Generation of Solutions

Different formulas for generating sequence of integer solutions based on the given solution are presented below:

Let $\left(x_0, y_{0,} z_0\right)$ be any given solution to (1)

Formula: 1

Let $\left(x_{1,} y_1, z_1\right)$ given by

$$x_1=x_0,\ y_1=y_0+2h,\ z_1=h-z_0 \qquad (12)$$

be the 2^{nd} solution to (1). Using (12) in (1) and simplifying, one obtains

$$h=12y_0+26z_0$$

In view of (12), the values of y_1 and z_1 are written in the matrix form as

$$\left(y_1, z_1\right)^t=M\left(y_0, z_0\right)^t$$

where

$M=\begin{pmatrix}25 & 52\\ 12 & 25\end{pmatrix}$ and t is the transpose

The repetition of the above process leads to the n^{th} solutions y_n, z_n given by

$$\left(y_n, z_n\right)^t=M^n\left(y_0, z_0\right)^t$$

If α, β are the distinct eigenvalues of M, then

$$\alpha=25+4\sqrt{39}\ ,\ \beta=25-4\sqrt{39}$$

We know that

$$M^n=\frac{a^n}{(\alpha-\beta)}(M-\beta I)+\frac{\beta^n}{(\beta-\alpha)}(M-\alpha I),\ I=2\times 2 \text{ Identity matrix}$$

Thus, the general formulas for integer solutions to (1) are given by

$$x_n=x_0$$

$$y_n=\left(\frac{\alpha^n+\beta^n}{2}\right)y_0+13\left(\frac{\alpha^n-\beta^n}{2\sqrt{39}}\right)z_0$$

$$z_n=3\left(\frac{\alpha^n-\beta^n}{2\sqrt{39}}\right)y_0+\left(\frac{\alpha^n+\beta^n}{2}\right)z_0$$

Formula: 2

Let $\left(x_1, y_1, z_1\right)$ given by

$$x_1=h-4x_0,\ y_1=h-4y_0,\ z_1=4z_0 \qquad (13)$$

be the 2^{nd} solution to (1). Using (13) in (1) and simplifying, one obtains

$h = 2x_0 + 6y_0$

In view of (13), the values of x_1 and y_1 are written in the matrix form as

$(x_1, y_1)^t = M(x_0, y_0)^t$

where

$M = \begin{pmatrix} -2 & 6 \\ 2 & 2 \end{pmatrix}$ and t is the transpose

The repetition of the above process leads to the n^{th} solutions x_n, y_n given by

$(x_n, y_n)^t = M^n (x_o, y_0)^t$

If α, β are the distinct eigenvalues of M, then

$\alpha = 4, \beta = -4$

Thus, the general formulas for integer solutions to (1) are given by

$x_n = 4^{n-1}((1+3(-1)^n)x_o + 3(1-(-1)^n)y_0)$

$y_n = 4^{n-1}(((1-(-1)^n)x_o + (3+(-1)^n)y_0)$

$z_n = 4^n z_o$

Formula: 3

Let $(x_1, y_1 z_1)$ given by

$x_1 = -3x_0 + 4h,\ y_1 = 3y_0,\quad z_1 = h + 3z_0$ (14)

be the 2^{nd} solution to (1). Using (14) in (1) and simplifying, one obtains

$h = 8x_0 + 26z_0$

In view of (14), the values of x_1 and z_1 are written in the matrix form as

$(x_1, z_1)^t = M(x_0, z_0)^t$

where

$M = \begin{pmatrix} 29 & 104 \\ 8 & 29 \end{pmatrix}$ and t is the transpose

The repetition of the above process leads to the n^{th} solutions x_n, z_n given by

$(x_n, z_n)^t = M^n (x_0, z_0)^t$

If α, β are the distinct eigen values of M, then

$\alpha = 29 + 8\sqrt{13},\ \beta = 29 - 8\sqrt{13}$

Thus, the general formulas for integer solutions to (1) are given by

$$x_n = \left(\frac{\alpha^n + \beta^n}{2}\right)x_0 + \sqrt{13}\left[\frac{\alpha^n - \beta^n}{2}\right]z_0$$

$$y_n = 3^n y_0$$

$$z_n = \frac{1}{2\sqrt{13}}(\alpha^n - \beta^n)x_0 + \left(\frac{\alpha^n + \beta^n}{2}\right)z_0$$

III. CONCLUSION:

In this paper, an attempt has been made to obtain non-zero distinct integer solutions to the ternary quadratic Diophantine equation $3y^2 = 13z^2 - x^2$ representing homogeneous cone. As there are varieties of cones, the readers may search for other forms of cones to obtain integer solutions for the corresponding cones.

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