International Journal of Research in Engineering and Science (IJRES)
ISSN (Online): 2320-9364, ISSN (Print): 2320-9356
www.ijres.org Volume 9 Issue 3 ǁ 2021 ǁ PP. 08-13

Mellin Transform of Mittag-Leffler Density and its Relationship with Some Special Functions

Hussam M. Gubara[1], Alshaikh.A.Shokeralla[2*]
[1]*Department of Mathematics – Faculty of Mathematical Sciences and Statistics – Al-Neelain University – Sudan*
[2]**Department of Mathematics, College of Science and Arts, Al-Baha University, Al-Makhwah, KSA*

Abstract
In this paper, the relationship between gamma function and Mittag-Leffler density function is investigated. The Mellin transform of Mittag-Leffler density function also is presented. The expected values of some functions also is written as Mellin transform of some special cases of Mittag-Leffler density function by using some properties of Mellin transform.

Keywords
Mellin transform, Mittag-Leffler density, Gamma function.

Date of Submission: 21-02-2021 Date of acceptance: 04-03-2021

I. INTRODUCTION

The Mellin transform was introduced by Finnish mathematician Robert Hjalmer Mellin [1]. The classical Mellin transform is connected with the two-sided Laplace transform, then the classical Mellin transform is a linear integral transform like Laplace transform [2-4]. The classical Mellin transform is denoted and defined by

$$f^*(p) = M\{f(y)\} = \int_0^\infty f(y)\ y^{p-1}\ dy \qquad (1.1)$$

Where p is the parameter of Mellin transform?
By substitution $y = e^{-t}$, we obtain the two-sided Laplace transform [5]

$$f^*(p) = \int_{-\infty}^{\infty} f(e^{-t})\ e^{-pt}\ dy \qquad (1.2)$$

The inverse Mellin transform is denoted and defined by

$$f(y) = M^{-1}\{f^*(p)\} = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} y^{-p}\ f^*(p)\ dp \qquad (1.3)$$

In mathematical methods, the integral transforms have many applications in ordinary differential equations (ODEs) and partial differential equations (PDEs) and fractional differential equations (FDEs) [6-11]. Fractional differential equation is a generalization of ordinary differential equation and order of fractional differential equation is a real order (fractional or natural order).
Mittag-Leffler function is an important function in field of fractional calculus. The fundamental Mittag-Leffler function was introduced by Gosta Mittag-Leffler [12]. It is a generalization of exponential function and it is denoted and defined by

$$E_a(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\,1 + a\,j\,)} \qquad (1.4)$$

Two-parameter and three-parameter Mittag-Leffler functions are generalization of $E_a(x)$. A 2-parameter Mittag-Leffler function is denoted and defined by

$$E_{a,b}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\,b + a\,j\,)} \qquad (1.5)$$

A 3-parameter Mittag-Leffler function is denoted and defined by

$$E_{a,b}^{c}(x) = \sum_{j=0}^{\infty} \frac{\Gamma(c+j)\ x^j}{j!\ \Gamma(c)\Gamma(\,b + a\,j\,)} \qquad (1.6)$$

Many people have introduced statistical densities depend on special forms of Mittag-Leffler function [13-20]. In 2011 Mathai has introduced a general statistical density function related with a 3-parameter Mittag-Leffler function [21]. The Mittag-Leffler statistical density function is denoted and defined by

$$\emptyset_{a,b}^{c}(z) = \frac{z^{a\,b-1}}{c^b\,\Gamma(b)} \sum_{j=0}^{\infty} \frac{\Gamma(\,b+j\,)\,(-1)^j\,z^{a\,j}}{j!\;\;c^j\;\;\Gamma(\,ab+aj\,)} \qquad (1.7)$$

Where $z \geq 0$, $R(a) > 0$, $R(b) > 0$ and $c > 0$

This paper is structured as follows: Section two represents some properties of Mellin transform and its relationship with some special functions. In section three we will find Mellin transform of special cases of Mittag-Leffler density.

II. SOME PROPERTIES OF MELLIN TRANSFORM

We know that all integral transforms are linear transforms, then

$$M\left\{\sum_{i=1}^{n} a_i\,f_i(y)\right\} = \int_0^{\infty} \sum_{i=1}^{n} a_i\,f_i(y)\;\;y^{p-1}\;\;dy = \sum_{i=1}^{n} ai\;\;\int_0^{\infty} f_i(y)\;\;y^{p-1}\;\;dy$$

$$= \sum_{i=1}^{n} ai\;\;M\{\,f(y)\,\}\;=\;\sum_{i=1}^{n} ai\;\;f_i^*(p) \qquad (2.1)$$

And Mellin transform of exponential function has a relationship with gamma function

$$M\{\,e^{-r\,y}\,\} = \int_0^{\infty} e^{-r\,y}\,y^{p-1}\;\;dy \qquad (2.2)$$

Where r is a positive real number?

By a change of variables $r\,y = t$ we obtain

$$M\{\,e^{-r\,y}\,\} = M\{\,e^{-\,t}\,\} = \int_0^{\infty} e^{-t}\,\left(\frac{t}{r}\right)^{p-1}\;\;\frac{dt}{r}$$

$$=\;r^{-p}\;\int_0^{\infty} e^{-\,t}\;t^{p-1}\;\;dt =\;r^{-p}\;\Gamma(p) \qquad (2.3)$$

We can find Mellin transform of $e^{-\,r\,y^2}$

$$M\left\{\,e^{-\,r\,y^2}\,\right\} = \int_0^{\infty} e^{-\,r\,y^2}\;y^{p-1}\;\;dy \qquad (2.4)$$

By a change of variables $t = r\,y^2 \quad \Rightarrow \quad y = \left(\frac{t}{r}\right)^{\frac{1}{2}}$, we get

$$M\left\{\,e^{-\,r\,y^2}\,\right\} = \int_0^{\infty} e^{-\,t}\;\left(\frac{t}{r}\right)^{\frac{p-1}{2}}\;\frac{1}{2}\left(\frac{1}{r\,t}\right)^{\frac{1}{2}}\;\;dt$$

$$=\frac{1}{2}\,r^{-p}\;\int_0^{\infty} e^{-\,t}\;t^{\frac{p}{2}-1}\;\;dt\;=\;\frac{1}{2}\,r^{-p}\,\Gamma\left(\frac{p}{2}\right) \qquad (2.5)$$

Mellin transform of first derivative is written as

$$M\left\{\frac{df}{dy}\right\} = \int_0^{\infty} \frac{df}{dy}\;y^{p-1}\;\;\;dy \qquad (2.6)$$

By using integration by parts, we get

$$M\left\{\frac{df}{dy}\right\} = [\,f(y)y^{p-1}]_0^{\infty} - \int_0^{\infty} (p-1)\,f(y)y^{p-2}\;\;\;dy \qquad (2.7)$$

if $\lim_{y\to\infty} f(y) = 0$, we get

$$M\left\{\frac{df}{dy}\right\} = -(p-1)\;\int_0^{\infty} f(y)y^{(p-1)\,-1}\,dy = -(p-1)\,f^*(p-1) \qquad (2.8)$$

From Mellin transform of exponential function we can deduce the Mellin transform of some trigonometric functions

$$M\{\,\sin ry\;\;\} = \int_0^{\infty} \sin ry\quad y^{p-1}\;\;dy$$

$$= \frac{1}{2i} \int_0^\infty (e^{r\,i\,y} - e^{-r\,i\,y}) \quad y^{p-1}\ dy \qquad (2.9)$$

From equation (2.3) , (2.6) becomes

$$\frac{1}{2i} [(-i\,r)^{-p} - (i\,r)^{-p}]\, \Gamma(p) = \frac{r^{-p}\, \Gamma(p)}{2\,i} [(-i)^{-p} - (i)^{-p}] \qquad (2.10)$$

by using polar forms of complex numbers $i = e^{\frac{\pi}{2}i}$, $-i = e^{-\frac{\pi}{2}i}$, we can rewrite (2.10) as

$$M\{ \sin ry \} = r^{-p}\, \Gamma(p) \left[\frac{\left(e^{-\frac{\pi}{2}i} \right)^{-p} - \left(e^{\frac{\pi}{2}i} \right)^{-p}}{2\,i} \right]$$

$$= r^{-p}\, \Gamma(p) \left[\frac{e^{\frac{\pi}{2}i\,p} - e^{-\frac{\pi}{2}i\,p}}{2\,i} \right] = r^{-p}\, \Gamma(p)\ \sin\frac{\pi}{2}p \qquad (2.11)$$

By using Mellin transform of $(\sin ry)$ and using Mellin transform of first derivative, we can deduce Mellin transform of $(\cos ry)$

$$M\{ \cos ry \} = M\left\{ \frac{d}{dy}\left(\frac{\sin ry}{r} \right) \right\} \qquad (2.12)$$

from (2.8) , if $f(y) = \frac{\sin ry}{r}$, then

$$M\{ \cos ry \} = -(p-1)\, f^*(p-1) \qquad (2.13)$$

by using equation (2.11) , we get

$$M\{ \cos ry \} = -(p-1)\ \frac{1}{r}\ r^{-(p-1)}\, \Gamma(p-1)\ \sin\left[\frac{\pi}{2}(p-1) \right]$$

$$= -\Gamma(p)\, r^{-p}\ \sin\left[\frac{\pi}{2}p - \frac{\pi}{2} \right] = -\Gamma(p)\, r^{-p} \left(-\cos\frac{\pi}{2}p \right)$$

$$= \Gamma(p)\, r^{-p}\ \cos\frac{\pi}{2}p \qquad (2.14)$$

In previous functions, we can see the strong relationship between Mellin transform of these functions and gamma function. Now we can find the relationship between beta function and Mellin transform of $\left(\frac{1}{1+y} \right)$

$$M\left\{ \frac{1}{1+y} \right\} = \int_0^\infty \frac{1}{1+y}\ y^{p-1}\ \ dy \qquad (2.15)$$

by a change of variables $y = \frac{t}{1-t} \quad \Rightarrow \quad dy = \frac{dt}{(1-t)^2}$

if $y = 0 \quad \Rightarrow \quad t = 0$, if $y \to \infty \quad \Rightarrow \quad t \to 1$, we get

$$M\left\{ \frac{1}{1+y} \right\} = \int_0^1 \frac{1}{1 + \frac{t}{1-t}}\ \frac{t^{p-1}}{(1-t)^{p-1}}\ \frac{1}{(1-t)^2}\ dt$$

$$= \int_0^1 (1-t)\ \frac{t^{p-1}}{(1-t)^{p-1}}\ \frac{1}{(1-t)^2}\ dt$$

$$= \int_0^1 (1-t)^{-p}\ t^{p-1}\ dt = \beta(1-p\,, p) = \frac{\Gamma(1-p)\, \Gamma(p)}{\Gamma(1-p+p)}$$

$$= \Gamma(1-p)\, \Gamma(p) = \frac{\pi}{\sin \pi p} \qquad (2.16)$$

III. MELLIN TRANSFORM OF SPECIAL CASES OF MITTAG-LEFFLER DENSITY

Mittag-Leffler density is very important function in statistical field. Special case of Mittag-Leffler density at $a = 1$ is equal to gamma density.

We can substitute $a = 1$ in equation (1.7) , we get

$$\emptyset_{1,b}^{c}(z) = \frac{z^{b-1}}{c^b\, \Gamma(b)} \sum_{j=0}^{\infty} \frac{\Gamma(b+j)\, (-1)^j\, z^j}{j!\ c^j\ \Gamma(b+j)}$$

$$= \frac{z^{b-1}}{c^b\, \Gamma(b)} \sum_{j=0}^{\infty} \frac{\left(-\frac{z}{c} \right)^j}{j!} = \frac{z^{b-1}}{c^b\, \Gamma(b)}\, e^{-\frac{z}{c}} \qquad (3.1)$$

This is the gamma density function with the parameters b and $\frac{1}{c}$, its integration from zero up to infinity is equal to one

$$\int_0^\infty \emptyset_{1,b}^{c}(z)\ dz = \frac{1}{c^b\,\Gamma(b)}\int_0^\infty z^{b-1}\,e^{-\frac{z}{c}}\ dz \qquad (3.2)$$

by substitution $z = c\,t$, we get

$$\int_0^\infty \emptyset_{1,b}^{c}(z)\ dz = \frac{1}{c^b\,\Gamma(b)}\int_0^\infty (c\,t)^{b-1}\,e^{-t}\ \ c\,dt$$

$$= \frac{1}{c^b\,\Gamma(b)}\int_0^\infty c^b\,t^{b-1}\,e^{-t}\ \ dt = \frac{c^b\,\Gamma(b)}{c^b\,\Gamma(b)} = 1 \qquad (3.3)$$

Expected value of any function $f(z)$ is denoted and defined by

$$E(\,f(z)\,) = \frac{1}{c^b\,\Gamma(b)}\int_0^\infty f(z)\,z^{b-1}\,e^{-\frac{z}{c}}\ dz \qquad (3.4)$$

if $f(z) = z^r$, we obtain

$$E(\,z^r\,) = \frac{1}{c^b\,\Gamma(b)}\int_0^\infty z^{r+b-1}\,e^{-\frac{z}{c}}\ dz \qquad (3.5)$$

by a change of variables $z = ct$, we get

$$E(\,z^r\,) = \frac{1}{c^b\,\Gamma(b)}\int_0^\infty (ct)^{r+b-1}\,e^{-t}\ \ c\,dt$$

$$\frac{c^{b+r}}{c^b\,\Gamma(b)}\int_0^\infty t^{r+b-1}\,e^{-t}\ \ dt = \frac{c^r\,\Gamma(b+r)}{\Gamma(b)} \qquad (3.6)$$

If $f(z) = e^{-r\,z}$ where $r + \frac{1}{c} > 0$, then

$$E(\,e^{-r\,z}\,) = \frac{1}{c^b\,\Gamma(b)}\int_0^\infty z^{b-1}\,e^{-\left(r+\frac{1}{c}\right)z}\ \ dz \qquad (3.7)$$

By substitution $t = \left(\frac{c\,r+1}{c}\right)z$, we get

$$E(\,e^{-r\,z}\,) = \frac{1}{c^b\,\Gamma(b)}\int_0^\infty \left(\frac{c\,t}{c\,r+1}\right)^{b-1}\ e^{-t}\ \frac{c}{c\,r+1}\ \ dt$$

$$= \frac{c^b}{(\,cr+1)^b\,c^b\,\Gamma(b)}\int_0^\infty t^{b-1}\,e^{-t}\ \ dt$$

$$= \frac{\Gamma(b)}{(\,cr+1)^b\,\Gamma(b)} = \frac{1}{(\,cr+1)^b} \qquad (3.8)$$

Then

$$E(\,e^{-r\,z}\,) = (\,cr+1)^{-\,b} \qquad (3.9)$$

We can prove (3.9) by using expansion of exponential function and using the properties of expected value We know that

$$e^{-r\,z} = \sum_{k=0}^{\infty}\frac{(\,-r\,z\,)^k}{k!} \qquad (3.10)$$

By taking the expected value for both sides of (3.10)

$$E(\,e^{-r\,z}\,) = E\left(\sum_{k=0}^{\infty}\frac{(\,-r\,z\,)^k}{k!}\right) \qquad (3.11)$$

Since the expected has the linear property, we can rewrite (3.11) as

$$E(\,e^{-r\,z}\,) = \sum_{k=0}^{\infty}\frac{(\,-r\,)^k}{k!}\,E(\,z^k\,) \qquad (3.12)$$

By using the equation (3.6) , then (3.12) becomes

$$E(e^{-rz}) = \sum_{k=0}^{\infty} \frac{(-r)^k}{k!} \frac{\Gamma(b+k)}{\Gamma(b)} c^k$$
$$= \sum_{k=0}^{\infty} \frac{(-c\,r)^k}{k!} \frac{\Gamma(b+k)}{\Gamma(b)}$$
$$= \sum_{k=0}^{\infty} \frac{(-c\,r)^k}{k!} \times (b+k-1) \times (b+k-2) \times \ldots \times b \qquad (3.13)$$

this is the left hand side of equation (3.9) .
By using binomial theory, then the right hand side of (3.9) is written as

$$(c\,r+1)^{-b} =$$
$$= 1 - \frac{b}{1!}\, cr + \frac{b\,(b+1)}{2!}\,(cr)^2 + \ldots$$
$$+ (-1)^k\,(cr)^k \times \frac{b(b+1) \times \ldots \times (b+k-1)}{k!} + \ldots$$
$$= \sum_{k=0}^{\infty} \frac{(-c\,r)^k}{k!} \times (b+k-1) \times (b+k-2) \times \ldots \times b$$
$$= \sum_{k=0}^{\infty} \frac{(-c\,r)^k}{k!} \frac{\Gamma(b+k)}{\Gamma(b)} = E(e^{-rz)} \qquad (3.14)$$

Then left hand side is equal to right hand side of equation (3.9)
Mellin transform of Mittag-Leffler density at $a = 1$ is written as

$$M\{\emptyset_{1,b}^{c}(z)\} = \frac{1}{c^b\,\Gamma(b)} \int_0^{\infty} z^{b-1}\, e^{-\frac{z}{c}} \quad z^{p-1}\, dz = E(z^{p-1}) \qquad (3.15)$$

From equation (3.6) , we get

$$M\{\emptyset_{1,b}^{c}(z)\} = E(z^{p-1}) = \frac{c^{p-1}\,\Gamma(b+p-1)}{\Gamma(b)} \qquad (3.16)$$

We can multiply and divide by $\Gamma(1-p)$, we can rewrite (3.16) as

$$M\{\emptyset_{1,b}^{c}(z)\} = \frac{\Gamma(b+p-1)\,\Gamma(1-p)}{\Gamma(b)} \frac{c^{p-1}}{\Gamma(1-p)}$$
$$= \beta(b+p-1, 1-p) \frac{c^{p-1}}{\Gamma(1-p)} \qquad (3.17)$$

If $= 1$, then

$$M\{\emptyset_{1,1}^{c}(z)\} = \beta(p, 1-p) \frac{c^{p-1}}{\Gamma(1-p)} = \frac{\pi}{\sin \pi p} \frac{c^{p-1}}{\Gamma(1-p)} \qquad (3.18)$$

From previous equations, we can remark that there is strong relationship between Mellin transform of Mittag-Leffler density and gamma and beta functions.

IV. CONCLUSION

The objective of this paper was to find the Mellin transform of special cases of Mittag-Leffler density function at $a = 1$ and its relation with some special function like gamma and beta functions and conclude that the Mellin transform of Mittag-Leffler density function at $a = 1$ is written as expected values of some functions.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

Acknowledgements

The author would like to appreciate the effort from editors and reviewers

REFERENCES

[1]. Butzer, P.L., Kilbas, A.A., Trujillo, J.J., Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, J. Math. Anal. Appl.,10(2001), 101-106

[2]. Butzer, P.L., Jansche, S., self-contained approaches to Mellin transform analysis for square integrable functions applications, Integral Transform. Spec. Funct., 8(1999), 175-198.

[3]. Butzer, P.L., Jansche, S., A direct approach to Mellin transform, J. Fourier Anal., 3(1997), 325-376.

[4]. Rooney, P. G., A survey of Mellin multipliers, Res. Notes Math., 138(1985), 176-187.

[5]. Flajolet, P., Gourdon, X., Dumas, P., Mellin transforms and asymptotics harmonic sums, Theoretical Computer Science, 144(1995), 3-58.

[6]. Mathai, A.M., Some properties of Mittag-leffler functions and matrix-variate analogues, Fractional Calculus and Applied Analysis, 13(2)(2010), 113-132.
[7]. Gorenflo, R., Kilbas, A. A., Mainardi, F., Rogosin, S. V., Mittag-Leffler Functions, Related Topics and Applications, Springer, New York, 2014.
[8]. Haubold, H. J., Mathai, A. M., Saxena, R. K., Mittag-Leffler functions and their applications, Journal of Applied Mathematics, ID298628(2011), 51pages.
[9]. Haubold, H. J., Mathai, A. M., An Introduction to Fractional Calculus, Mathematics Research Development, New York, 2017.
[10]. Mathai, A. M., A pathway to matrix-variate gamma and normal densities, Linear Algebra and its Application, 396(2005), 317-328.
[11]. Pillai, R.N., On Mittag-Leffler functions and related distributions, Ann. Inst. Statist. Math., 42(1990). 157-161.
[12]. Mittag-Leffler, G. M., Sur la nouvelle fonction $E_\alpha(x)$, Comptes Rendus de I'Academie des Sciences, vol.137(1903), 554-558
[13]. Nair, S. S., Statistical Distributions Connected with Pathway Model and Generalized Fractional Integral Operator, Ph.D Thesis, Banaras Hindu University, 2012.
[14]. Capelas de Oliveira, E., Mainardi, F., Vaz Jr, J., Models based on Mittag-Leffler functions for anomalous relaxation in dielectrics, The European Physical Journal, Special Topics 193(2011), 161-171.
[15]. Djrbashian, M.M., Bagian, A., On integral representations and measures associated with Mittag-Leffler type functions, Izv. Akdd. Nauk Armjanskvy SRR, Mathematika 10(1975), 483-508.
[16]. Gorenflo, R., Luchko, Yu., Mainardi, F., Analytical properties and applications of the Wright function, 2 No 4(1999) 383-414.
[17]. Hilfer, R., Seybold, H.J., Computation of the generalizated Mittag-Leffler function and its inverse in the complex plane, Integral Transforms and Special Functions, 17No. 9(2006), 637-652.
[18]. Kilbas, A.A., Saigo, M., On Mittag-Leffler type functions, fractional calculus operators and solution of integral equations, Integral Transforms and Special Functions, 4(1996), 355-370.
[19]. Bell, W. W., Special functions for Scientists and Engineers, D. Van Nostrand Company LTD, London,1967
[20]. Bender, C. M., Orszag, S. A., Advanced Mathematical Methods for Scientists and Engineers,1st-Springer, (1999).
[21]. Mathai, A. M., Haubold, H. J., A pathway from Bayesian statistical analysis to super statistics, Applied Mathematics and Computations, 218(2011), 799-804.