New Escape Criteria for Complex Fractals Generation in Jungck-Cr Orbit

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ABSTRACT: In recent years, researchers have studied the use of different iteration processes from fixed point theory for the generation of complex fractals. Examples are the Mann, the Ishikawa, the Noor, the Jungck-Mann and the Jungck-Ishikawa iterations. In this paper, we present a generalization of complex fractals, namely Mandelbrot, Julia and multicorn sets, using the Jungck-CR implicit iteration scheme. This type of iteration does not reduce to any of the other iterations previously used in the study of complex fractals; thus, this generalization gives rise to new fractal forms. We prove a new escape criterion for a polynomial of the following form and present some graphical examples of the obtained complex fractals.

Key words: Julia set; Mandelbrot set; Jungck-CR orbit; escape criteria.

I. INTRODUCTION

A very common method of generating fractal patterns in the complex plane is the repeated iteration of a complex function \( f : \mathbb{C} \rightarrow \mathbb{C} \) in the following way:

\[
z_{n+1} = f(z_n),
\]

where \( f \) depends on some constant \( c \in \mathbb{C} \) and \( z_c \in \mathbb{C} \) is the point from a consider area of the complex plane [38]. The two most famous examples of such fractals are the Mandelbrot and Julia sets.

The Mandelbrot and Julia sets are some of the best known illustrations of a highly complicated chaotic system generated by a very simple mathematical process. They were introduced by Benoit Mandelbrot in the late 1970s [27], but Julia sets were studied much earlier, namely in the early twentieth century by French mathematicians Pierre Fatou [10] and Gaston Julia [17]. While working at IBM, Mandelbrot studied their works and plotted the Julia sets for \( z^2 + c \). Building on these results, he plotted the Mandelbrot set. He was surprised by his results. Since then, many mathematicians have studied different properties of the Mandelbrot and Julia sets and proposed accordingly various generalizations. The first and most obvious generalization was the use of the \( z^2 + c \) function instead of the quadratic one used by Mandelbrot [8, 26]. Further, additional types of functions were studied: rational [32], transcendental [9], elliptic [24], anti-polynomials [6] etc. Another development, building on the study of the Mandelbrot and Julia sets, was the extension from complex numbers to other algebras, e.g. quaternions [7], octonions [15], bicomplex numbers [23] etc.

In recent years, some researchers have focused on a different kind of generalization. They used results that can be found in fixed point theory. In this theory, we can find methods of locating fixed points that replace the feedback process (1) with other types of iteration processes. The use of different iteration processes began in 2004 in the works of Rani and Kumar [36, 37]. They used the Mann iteration and proved new escape criteria for the generation of the Mandelbrot and Julia sets using this type of iteration. Further studies on the use of different iteration processes were conducted, focusing on two in particular. The first type, like the Mann iteration used by Rani and Kumar, was an explicit process. Examples of processes of this type that were used to generate the...
Mandelbrot and Julia sets are the following: Ishikawa [5, 14], Noor [2], S [22] and Abbas [25]. The second type of iteration process is an implicit one. In the literature, we can find the use of the Jungck-Mann [28], Jungck-Ishikawa [28] and Jungck-Noor [21] iterations, which are examples of such processes. Moreover, studies on the noise-perturbed versions of the Mandelbrot and Julia sets generated by the different iteration processes were conducted [1, 29, 35].

The Mandelbrot and Julia sets are not only interesting from a mathematical point of view. They have applications in other fields also, e.g. physics [3], biology [4] and robotics [45]. One of the most natural applications of the Mandelbrot and Julia sets – because of their beauty – was their use in computer graphics. The sets were used as a source of aesthetic patterns [44, 46], for creating realistic phenomena and landscapes [11] or for image manipulation [40]. In [42], Sun et al. have used Julia sets as the base for creating a dictionary with domain blocks used in fractal image compression. Moreover, in [41], Sun et al. have proposed an image encryption algorithm in which Julia set parameters are used to generate a random sequence as the initial keys for the algorithm.

### II. PRELIMINARIES

In this section, we provide the preliminary definitions that form the basis of our work.

#### Definition 3.1: (Multicorn) Let $A_\xi(z) = z^a + \xi$, where $\xi \in \mathbb{C}$. The multicorn $M^*$ for $A_\xi$ is defined as the collection of all $\xi \in \mathbb{C}$ to which the orbit of 0 under the action of $A_\xi$ is bounded, i.e.

$$M^* = \{\xi \in \mathbb{C} : |A_{\xi}^n(0)| \leq M, \quad n \geq 0\}.$$
The most widely used algorithm to generate images of Julia, Mandelbrot and multicorn sets is the escape time algorithm. The colour of each point in the algorithm is determined based on the number of iterations necessary to evaluate whether the orbit sequence tends to infinity or not. In order to establish whether the orbit escapes or not we use the escape criterion. For instance, for the classical Mandelbrot and Julia sets, i.e. sets defined by

$$Q_\zeta(z) = z^2 + \zeta,$$

the escape criterion is the following:

If there exists \( k = 0 \) such that

$$|Q_\zeta^k(z)| > \max(|\zeta|, 2),$$

then \( Q_\zeta^n(z) \) escapes as \( n \) goes to infinity.

We call the right side of (6) the escape threshold (or bailout value). This threshold can be different for different functions \( Q_\zeta \) and plays a very important role in the generation of Mandelbrot and Julia sets.

The escape time algorithms for the generation of Mandelbrot and Julia sets are presented in Algorithm 1 and 2 respectively. For the generation of multicorns, we can use the same algorithm as for the Mandelbrot set, but with \( A_\zeta \) instead of \( Q_\zeta \) and the critical point set to 0.

In fixed point theory, there are many theorems and methods that allow one to find fixed points in a given mapping. One of the main foci of the theory is the iterative approximation of fixed points. For this, we use different kinds of iteration processes. Let us recall some of them.

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Algorithm 1 Mandelbrot set generation

Require: \( q : C \rightarrow C \) – polynomial function, \( A \) – area, \( K \) – the maximum number of iterations, \( \text{colormap}[0..C-1] \) – colormap with \( C \) colours.

Ensure: Mandelbrot set for the area \( A \).

1. \( \text{for } c \in A \text{ do} \)
2. \( \quad R = \text{calculate escape threshold} \)
3. \( \quad n = 0 \)
4. \( \quad z_0 = \text{critical point of } q \)
5. \( \quad \text{while } n < K \text{ do} \)
6. \( \quad \quad z_{n+1} = q(z_n) \)
7. \( \quad \quad \text{if } |z_{n+1}| > R \text{ then} \)
8. \( \quad \quad \quad \text{break} \)
9. \( \quad \quad \quad \text{end if} \)
10. \( \quad \quad n = n + 1 \)
11. \( \quad \text{end while} \)
12. \( \quad i = (C - 1) \times \frac{n}{K} \)
13. \( \quad \text{colourtz} \) with \( \text{colormap}[i] \)
14. \( \text{end for} \)

Algorithm 2 Julia set generation

Require: \( q : C \rightarrow C \) – polynomial function, \( c \in C \) – parameter, \( A \) – area, \( K \) – the maximum number of iterations, \( \text{colormap}[0..C-1] \) – colormap with \( C \) colours.

Ensure: Julia set for the area \( A \).

1. \( R = \text{calculate escape threshold} \)
2. \( \text{for } z \in A \text{ do} \)
3. \( \quad n = 0 \)
4. \( \quad \text{while } n < K \text{ do} \)
5. \( \quad \quad z_{n+1} = q(z_n) \)
6. \( \quad \quad \text{if } |z_{n+1}| > R \text{ then} \)
7. \( \quad \quad \quad \text{break} \)
8. \( \quad \quad \quad \text{end if} \)
9. \( \quad \quad n = n + 1 \)
10. \( \quad \text{end while} \)
11. \( \quad i = (C - 1) \times \frac{n}{K} \)
12. \( \quad \text{colourtz} \) with \( \text{colormap}[i] \)
13. \( \text{end for} \)

Definition 4 – (Picard iteration [33]) Let \( T : X \rightarrow X \) be a mapping on a metric space \( (X, d) \), where \( d \) is a metric and let \( x_0 \in X \) be a starting point. The Picard iteration is defined as follows:

\[ x_{n+1} = T(x_n), \quad n = 0, 1, 2, \ldots \]  

(7)

The sequence \( \{x_n\} \) is called the Picard orbit of \( x_0 \).
Let us notice that the Jungck-Noor iteration reduces to the Jungck-Ishikawa iteration if $\alpha = 0$. This is the most general iteration of the above.

Let us notice that the Jungck-Noor iteration reduces to the Jungck-Ishikawa iteration if $\beta = 0$. This is the most general iteration of the above.
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Let us notice that the Picard iteration is the iteration used in the generation of complex fractals. In the literature, we can find studies that have replaced the Picard iteration with other iterations, e.g. with the ones presented in Section 2 [21, 28]. In this section, we show how to use one of the implicit iterations, namely the Jungck-CR iteration, for the generation of Julia, Mandelbrot and multicorn sets.

Let us start with the definition of a Jungck-CR iteration and its orbit.

Definition 9—(Jungck-CR iteration [16]). Let \( S, T : X \rightarrow X \) be mappings on a metric space \( (X, d) \), where \( \text{d} \) is a metric and let \( x_0 \in X \) be a starting point. The Jungck-CR iteration is defined as follows:

\[
S(x_{n-1}) = (1 - a) S(y_n) + a T(z_n),
\]

\[
S(y_n) = (1 - \beta) T(x_n) = \beta T(z_n),
\]

\[
S(z_n) = (1 - \gamma) S(x_n) + \gamma T(y_n),
\]

where \( a, \beta, \gamma \in [0, 1] \). The sequence \( \{x_n\} \) is called the Jungck-CR orbit of \( x_0 \).

Let us notice that the Jungck-CR iteration does not reduce to any of the discussed iterations: Picard, Jungck-Mann, Jungck-Ishikawa, Jungck-Noor. Thus, using this iteration creates a completely new orbit and, by consequence, new fractal sets. In the Picard orbit we use only one mapping and in Jungck-CR we have two mappings. Thus, if we want to replace the Picard orbit with the Jungck-CR orbit, we need to take into account the different number of mappings in the iterations. We handle this in the following way. Let

In the Picard orbit we use only one mapping and in Jungck-CR we have two mappings. Thus, if we want to replace the Picard orbit with the Jungck-CR orbit, we need to take into account the different number of mappings in the iterations. We handle this in the following way. Let \( Q_{z} : \mathbb{C} \rightarrow \mathbb{C} \) be a polynomial function. We deconstruct \( Q_{z} \) into two mappings \( S, T \) in such a way that \( Q_{z} = T \cdot S \) and \( S \) is injective. In the case of multicons, we deconstruct \( Q_{z} \) in the following way:

\( Q_{z} = T \cdot S \), where \( T = A_{z} \) and \( S \) is injective. Of course, this type of deconstruction restricts the choice of the polynomial functions that can be used. With the deconstruction, we also need to derive a new escape criterion for the mappings (1).
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\[ 2^{\alpha \cdot |z|} = 1, \text{ where } \alpha, \beta, (0, 1) \text{ and define } \{ z_n \} \subset N \text{ as follows:} \]

\[
S(z_{n+1}) = (1 - \alpha) S(z_n) - \alpha T(y_n),
S(y_{n}) = (1 - \beta) T(z_n) + \beta T(y_n),
S(y_{n}) = (1 - \beta) T(z_n) + \beta T(y_n),
\]

\[ n = 0, 1, 2, \ldots, \]

where \[ z_0 = z \text{. Then } |z_n| \leq \frac{8}{5} \text{ if } |z| \leq 1. \]

Proof: Because \[ T(z) = z + c, S(z) = az \text{ and } z_0 = z, \]

\[
|S(y_0)| = |(1 - \beta) T(z) + \beta T(y_0)| = |(1 - \beta)(z + c) + \beta(y_0 + c)|
= |(1 - \beta) z + \beta c| + |c| = |z| + |c| = |z| - |z - c|
\]

(because \[ |z| = |c| \text{ and } = 1). \]

The above expression gives us

\[
|au_c| = |z| - |z - c| = |z| (|z| \leq 1) - (1 + |c|).
\]

Thus

\[
|u_c| = |z| - |z - c| \leq 1 - (1 + |c|).
\]

In the second step of the iteration we have

\[
|S(y_0)| = |(1 - \beta) T(z) + \beta T(y_0)| = |(1 - \beta)(z + c) + \beta(y_0 + c)|
= |(1 - \beta) z + \beta c| + |c| = |z| + |c| = |z| - |c|.
\]

Since \[ |z| > 2^{\alpha \cdot |z|} = 1 \text{, which implies } |z|^{\alpha \cdot |z|} \leq 1 - 1 \leq |z|, \text{ hence } |u_0| \leq |z| \leq |z| < \infty. \]

We get

\[
|au_0| = (\beta - \beta) |z| - |c| = \beta |z| - |c| = (\beta - \beta) |z| - |c| = (\beta - \beta) |z| - (1 + |c|)|/|z| = (\beta - \beta) |z| - (1 + |c|)|/|z| = (\beta - \beta) |z| - (1 + |c|). \]

Thus

\[
|y_0| = |z| \beta |z| \leq 1 - (1 + |c|).
\]

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Since $|z| = |c| > 2^{1+|a|} \beta^{-1}$ and $|z| = |c| > 2^{1+|a|} \beta^{-1}$, so $|z| = 2^{1+|a|} \beta^{-1}$, thus

$|\beta| = 1 = |z| = 1 + |a|$. Hence

$|\beta| \approx 1 = |z| \approx 1 + |a| \approx 1$. Thus

In the third step of the iteration we have

$|S(z_1)| = |(1 - \alpha)S(y_0) + \alpha T(z_0)| = |(1 - \alpha)axy_0 + \alpha (y_0 + c)|$

$= |(1 - \alpha)axy_0 + \alpha y_0 + \alpha c| = \alpha y_0 - (1 - \alpha)ax_0 + \alpha c|

= \alpha y_0 - |c||y_0| + \alpha |c| (\text{because } |y_0| = |c|)

= |\alpha y_0 - (1 - |a|).|

Moreover, $|S(z_1)| = \alpha y_0$. Thus

$|z_1| = |y_0| \alpha y_0 - (1 - |a|).$

Because $|y_0| = |z|$ and $|z| = 2^{1+|a|} \beta^{-1}$, thus $\alpha y_0 - (1 - |a|) > 2$. Therefore,

$|z_1| = |y_0| \alpha y_0 - (1 - |a|)$. By consequence

$|z_1| = |z| \alpha y_0 - (1 - |a|).$

Since $|z| = 2^{1+|a|} \beta^{-1}$ and $|z| > 2^{1+|a|} \beta^{-1}$, and $|z| = 2^{1+|a|} \beta^{-1}$, so $|z| = 2^{1+|a|} \beta^{-1}$.

By consequence, $\alpha y_0 - (1 - |a|) > 1$. Therefore, there exists $> 0$ such that $\alpha y_0 - (1 - |a|) > 1 > 1$. Consequently, $|z_1| > (1 + \beta)|z|$. In particular, $|z_1| > |z|$. Thus, we may apply the same argument repeatedly to find $|z_2| > (1 + \beta)|z_1|$. This way, the orbit of $z$ tends to infinity, which completes the proof.

Corollary 1 — Suppose that

$|c| > 2(1 + |a|) \beta^{-1}$, $|c| = 2(1 + |a|) \beta^{-1}$, and $|c| > 2(1 + |a|) \beta^{-1}$,

then the Jungck-CR orbit escapes to infinity.
**Corollary 2.** (Escape Criterion). Let $a, b, c, (0, 1)$ and
\[
|z| > \max \, |c|, \quad 2(1 + |a|)^{\beta}, \quad 2(1 + |a|)^{\gamma}, \quad 2(1 + |a|)^{\delta}
\]
then there exists $\delta > 0$, such that $|z_0| > (1 + \delta)|z|$ and $|z_0| \rightarrow \delta$ as $n \rightarrow \delta$.

**Corollary 3.** Suppose that
\[
|z_0| > \max \, |c|, \quad 2(1 + |a|)^{\beta}, \quad 2(1 + |a|)^{\gamma}, \quad 2(1 + |a|)^{\delta}
\]
for some $k = 0$. Then, there exists $\delta > 0$, such that $|z_k| > (1 + \delta)|z_k|$ and $|z_k| \rightarrow \delta$ as $n \rightarrow \delta$.

**IV. EXAMPLE**

In this section, we present some examples of the Julia, Mandelbrot and multicorn sets images obtained using the Jungck-CR orbit and the escape criterion derived in Section 3. The images were generated using the escape time algorithm that was implemented in Mathematica. The times required for generating the images were between 5 and 7 seconds on a computer with an Intel Core i5 (2 GHz) processor and 3 GB of RAM.

Examples of Julia sets in Jungck-CR orbit

The first example presents Julia sets for $Q_c(z) = z^3 + (3/2)z + c$ and $c = 1.75 + i$ generated using Jungck and Jungck-CR iterations. The obtained images are presented in Figure 1 and the parameters used to generate them were the following:

(a) $A = [-1.7, 1.7]^2$, $K = 50$, Jungck iteration.

(b) $A = [-1.5, 1.5] \times [-2.5, 2.5]$, $K = 50$, Jungck-CR iteration with $a = b = -0.9$.

(c) $A = [-2.5, 2.5]^2$, $K = 50$, Jungck-CR iteration with $a = 0.9$, $b = 0.5$, $\epsilon = 0.1$.

(d) $A = [-3.5, 3.5] \times [-6, 6]$, $K = 50$, Jungck-CR iteration with $a = b = -0.1$.

Figure 1: Julia set for $Q_c(z) = z^3 + (3/2)z + c$ and $c = 1.75 + i$ in (a) Jungck and (b)-(d) Jungck-CR orbit.
From the obtained images we can observe that the Julia sets generated using the Jungck-CR iteration differ in a significant way from the one obtained with the Jungck iteration. The shape of the set in Figure (b) is somewhat similar to the one from Figure (a), but the shapes of the sets from Figure (c) and (d) are completely different. The change of the parameters in the Jungck-CR iteration allows us to obtain very different shapes of Julia sets for the same polynomial used to generate them. Moreover, when we look at the area that the sets occupy, we see that the sets obtained with the Jungck-CR iteration occupy a larger area than the set generated using the Jungck iteration. In the second example we present various Julia sets generated using the Jungck-CR iteration. In order to generate the images of Julia sets from Figure 2 the following parameters were used:

(a) $Q(\varepsilon) = \varepsilon^2 + 2 + \varepsilon, \ \varepsilon = 17.28 + 5.5i, \ A = [-28, 14] \times [-28, 28], \ K = 50, \text{Jungck-CR iteration with } a = \beta = 0.1.$

(b) $Q(\varepsilon) = \varepsilon^3 + (3/2) + \varepsilon, \ \varepsilon = 0.72, \ A = [-2.3, 2.3], \ K = 50, \text{Jungck-CR iteration with } a = \beta = 0.5.$

(c) $Q(\varepsilon) = \varepsilon^4 + 3 + \varepsilon, \ \varepsilon = 3.9 + 7i, \ A = [-2.5, 2.5], \ K = 50, \text{Jungck-CR iteration with } a = \beta = 0.5.$

From the images, we see that using different combinations of the parameters, e.g. polynomial, parameters in the Jungck-CR iteration etc., we are able to obtain very diverse fractal patterns that have potential artistic applications.

4.2 Examples of Mandelbrot sets in the Jungck-CR orbit We begin with an example presenting the use of Jungck and Jungck-CR iterations. In the example,
we use $Q(z) = z^4 + 3z + c$. The parameters used to generate the images from Figure 3 were the following:

(a) $A = [-7, 6] \times [-6, 6], K = 20$, Jungck iteration.

(b) $A = [-8, 8], K = 20$, Jungck-CR iteration with $g = 0.5, \beta = 0.9, \alpha = 0.9$.

(c) $A = [-13, 10] \times [-11, 11], K = 20$, Jungck-CR iteration with $g = 0.9, \beta = 0.9, \alpha = 0.1$.

(d) $A = [-63, 60] \times [-61, 61], K = 20$, Jungck-CR iteration with $g = \beta = 0.1$.

Looking at the images in Figure 3 we can observe that the shape of the Mandelbrot set is different in the Jungck-CR orbit and the Jungck orbit. The number of bulbs in all the images is the same – three – but their shape changes with the change of the parameters in the Jungck-CR iteration. We can also observe that for different values of the $a$, $\beta$ and parameters the three-fold symmetry of the sets remains unchanged. Moreover, the area that the set occupies changes for different values of the parameters used in the Jungck-CR iteration.

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1. Looking at the images in Figure 3 we can observe that the shape of the Mandelbrot set is different in the Jungck-CR orbit and the Jungck orbit. The number of bulbs in all the images is the same – three – but their shape changes with the change of the parameters in the Jungck-CR iteration. We can also observe that for different values of the $a$, $\beta$ and parameters the three-fold symmetry of the sets remains unchanged. Moreover, the area that the set occupies changes for different values of the parameters used in the Jungck-CR iteration.

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Figure 3: Mandelbrot set for $Q(z) = z^4 + 3z + c$ in (a) Jungck and (b)-(d) Jungck-CR orbit.
Figure 4 presents examples of various Mandelbrot sets generated using the Jungck-CR iteration and the following parameters:

(a) $Q(x) = x^2 + 2x + c, A = [-35.5, 12] \times [-12, 12], K = 20$, Jungck-CR iteration with $a = 0.6, b = 0.2$.  

(b) $Q(x) = x^3 - (3/2)x + c, A = [-3.5, 3.5] \times [-5.5, 5.5], K = 20$, Jungck-CR iteration with $a = 0.1, b = 0.5, c = 0.7$.  

(c) $Q(x) = x^5 + 3x + c, A = [-12, 12], K = 20$, Jungck-CR iteration with $a = b = c = 0.5$.  

Similar to the case of the Julia sets, we can observe from the presented images that using the Jungck-CR iteration we are able to obtain various shapes of the corresponding Mandelbrot sets. Moreover, the change of the $a, b$ and parameters in the Jungck-CR iteration has great impact on the shape of the resulting set.
4.3 Examples of multicorns in the Jungck-CR orbit

Similar to the case of the Julia and Mandelbrot sets, we begin with an example of multicorns generated with the Jungck and Jungck-CR iterations. In the example, we present multicorns for \( Q_c^k(z) = z^2 + \alpha \beta + \gamma \). The obtained images of multicorns are presented in Figure 5. For their generation, the following parameters were used:

(a) \( A = [-7, 6] \times [-6, 6], K = 20, \text{ Jungck iteration.} \)

(b) \( A = [-8, 8]^2, K = 20, \text{ Jungck-CR iteration with } \alpha = \beta = -0.9. \)

(c) \( A = [-15, 10] \times [-11, 11], K = 20, \text{ Jungck-CR iteration with } \alpha = \beta = -0.5. \)

(d) \( A = [-100, 60] \times [-61, 61], K = 20, \text{ Jungck-CR iteration with } \alpha = \beta = -0.1. \)

From the obtained images in Figure 5, we can observe that the overall shapes of the sets obtained in the Jungck-CR orbit are completely different than the shape obtained in the Jungck orbit. The shape changes in a significant way with the change of the \( a, \beta \) and parameters of the Jungck-CR iteration. Moreover, the set in the Jungck orbit has a 5-fold symmetry, whereas the sets in the Jungck-CR orbit lose this type of symmetry. They only exhibit a mirror symmetry with the real axis as the line of symmetry. When we look at the area occupied by the sets, we observe that also in this case depending on the values of the parameters used in the Jungck-CR iteration the sets occupy different areas – for some values they occupy a smaller area and for others a larger.

Figures 5: Multicorn set for

(a) (b)

(c) (d)
In the second example, we present various multicorn sets generated using the Jungck-CR iteration. In order to generate the images of the sets from Figure 6 the following parameters were used:

\[
(a) \quad \mathcal{Q}_s^*(z) = z^2 + z + c, \quad \mathcal{A} = [-16.5, 8.5] \times [-8, 8], \quad K = 20, \quad \text{Jungck-CR iteration with } \alpha = 0.6, \beta = 0.7, \gamma = 0.5.
\]

\[
(b) \quad \mathcal{Q}_s^*(z) = z^2 + (3/2)z + c, \quad \mathcal{A} = [-2.5, 2.5] \times [-3.5, 3.5], \quad K = 20, \quad \text{Jungck-CR iteration with } \alpha = 0.9, \beta = 0.7, \gamma = 0.7.
\]

When we look at the images, we can observe that the shapes of the obtained sets are less interesting in comparison to the Mandelbrot and Julia sets. Moreover, we see that both sets have axial symmetries. The image in Figure 6(a) has one axis of symmetry, namely the real axis, whereas the image in Figure 6(b) has two axes of symmetry – the real and imaginary axis.

V. CONCLUSIONS

In this paper, the Jungck-CR iteration has been introduced in the study of complex fractals (Julia sets, Mandelbrot sets, multicorn). The new escape criterion for the Jungck-CR iteration has been established for the \(m\)-th degree complex polynomial functions. The Jungck-CR does not reduce to the Picard, Jungck-Mann, Jungck-Ishikawa, Jungck-Noor, nor any other iteration studied in the literature on the generation of complex fractals. Thus, the results of this paper open up a new class of complex fractals. Moreover, the obtained complex fractals could further extend the capabilities of the algorithms that use Mandelbrot and Julia sets, e.g. they can expand the domain dictionary used in fractal image compression [42] or broaden the space for the initial keys used in image encryption [41]. In future work, we will attempt to introduce the Jungck-CR and other implicit iteration processes into other types of complex fractals, e.g. fractals obtained through polynomiography.

REFERENCES

NEW ESCAPE CRITERIA FOR COMPLEX FRACTALS 1301


[17]. G. Julia, M‘emoire sur l’itéération des fonctions rationnelles, Journal de Mathématiques Pures et Appliquées , 8(1) (1918), 47-246.


