Extension of Some Common Fixed Point Theorems using Compatible Mappings in Fuzzy Metric Space

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Abstract: In this paper we have proved some common Fixed Point theorems for four mappings using the notion of compatibility.

Keywords: Fuzzy Metric Space, Compatible Mappings

I. Introduction

The concept of Fuzzy sets was investigated by Zadeh [1]. Here we are dealing with the fuzzy metric space defined by Kramosil and Michalek [2] and modified by George and Veeramani [3]. Grabiec [4] has also proved fixed point results for fuzzy metric space with different mappings. Singh and Chauhan[5] gave the results using the concept of compatible mappings in Fuzzy metric space. Jungek [6] introduced the concept of compatible mapping of type (A) and type (B). In fuzzy metric space. Singh and jain [7] proved the fixed point theorems in fuzzy metric space using the concept of compatibility and semicompatibility. Sharma[8] also done work on compatible mappings.

II. FUZZY METRIC SPACE

Definition[2]: A 3-tuple (X,M, *) is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^{2} \times [0, \infty]$ satisfying the following conditions

- (f1) $M(x, y, t) > 0$
- (f2) $M(x, y, t) = 1$ if and only if $x = y$
- (f3) $M(x, y, t) = M(y, x, t)$;
- (f4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
- (f5) $M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous.

Then M is called a fuzzy metric on X. Then $M(x, y, t)$ denotes the degree i.e. of nearness between x and y with respect to t.

Compatible and Non compatible mappings: Let A and S be mapping from a fuzzy metric space $(X,M, *)$ into itself. Then the mappings are said to be compatible if

$$\lim_{n \to \infty} M(ASx_{n}, SAx_{n}, t) = 1, \forall t > 0,$$

whenever $\{x_{n}\}$ is a sequence in X such that

$$\lim_{n \to \infty} Ax_{n} = \lim_{n \to \infty} x \in X$$

from the above definition it is inferred that A and S are non compatible maps from a fuzzy metric space $(X,M, *)$ into itself if

$$\lim_{n \to \infty} Ax_{n} = \lim_{n \to \infty} Sx_{n} = x \in X$$

but either

$$\lim_{n \to \infty} M(ASx_{n}, SAx_{n}, t) \neq 1,$$

or the limit does not exist.

Main Results:

Theorem: Let A,B,S,T be self maps of complete fuzzy metric space $(X,M, *)$ such that $a*b = \min(a,b)$ for some $y$ in X.

(a) $A(X) \subset T(X), B(X) \subset S(X), T(Y) \subset A(Y)$
(b) S and T are continuous.
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(c) \([A,S],[B,T]\) are compatible pairs of maps

(d) For all \(x, y \in X, k \in (0,1), t > 0\),

\[ M(AX, BY, KT) \geq \min \{ M(Sx, Ty, t), M(AX, SX, t), M(BY, Ty, t), M(BY, SX, t), M(AX, TY, t), M(Ay, Tx, t) \} \]

For all \(x, y \in X\), \(n \rightarrow \infty\) \(M(x, y, t) \rightarrow 1\) then \(A, B, S, T\) have a common fixed point in \(X\).

Proof: Let \(x_0\) be an arbitrary point in \(X\). Construct a sequence \(\{y_n\}\) in \(X\) such that \(y_{2n} = Tx_{2n-1} = Ax_{2n-2}\) and \(y_{2n} = Sx_{2n-1} = Tx_{2n-1}\) for \(n = 0, 1, 2, \ldots\)

Put \(x = x_{2n}, y = x_{2n+1}\),

\[ M(y_{2n+1}, y_{2n+2}, kt) = M(AX_{2n+1}Bx_{2n+1}, kt) \]

\[ \geq \min \{ M(Sx_{2n}, Tx_{2n+1}, t), M(AX_{2n}Sx_{2n+1}, M(Bx_{2n+1}, Tx_{2n+1}, t), M(Tx_{2n}, Ax_{2n+1}, t), M(AX_{2n}, Tx_{2n+1}, t), M(Bx_{2n+1}, Sx_{2n+1}, t) \} \]

\[ \geq \min \{ M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+1}, y_{2n+2}, t), 1 \} \]

Which implies

\[ M(y_{2n+1}, y_{2n+2}, t) \geq M(y_{2n+1}, y_{2n+2}, t) \]

In general

\[ M(y_{2n+1}, y_{2n+2}, t) \geq M(y_{2n+1}, y_{2n+2}, t) \]

(1)

To prove that \(\{y_n\}\) is a Cauchy sequence we will prove (b) is true for all \(n \geq n_0\) and every \(m \in N\)

\[ M(y_n, y_{n+m}, t) > 1 - \lambda \]

(2)

Here we use induction method

From (1) we have

\[ M(y_{2n+1}, y_{2n+2}, t) \geq M(y_{2n+1}, y_{2n+2}, t) \geq \ldots \geq M(y_0, y_1, t/k^2) \rightarrow 1 \text{ as } n \rightarrow \infty \]

i.e for \(t > 0\), \(\lambda \in (0,1)\). We can choose \(n_0 \in N\), such that

\[ M(y_{n_0}, y_{n_0+m}, t) > 1 - \lambda \]

(3)

Thus (2) is true for \(m = 1\). Suppose (2) is true for \(m = n\) then will show that it is true for \(m+1\). By the definition of fuzzy metric space, we have

\[ M(y_{n+1}, y_{n+m+1}, t) \geq \min \{ M(y_{n+1}, y_n, t), M(y_{n+m}, y_{n+m+1}, t) \} > 1 - \lambda \]

Hence (2) is true for \(m+1\). Thus \(\{y_n\}\) is a Cauchy sequence. By completeness of \((X, M, *)\), \(\{y_n\}\) converges (3)

Using (3), we have \(M(SX_{2n}, SX_{2n}, t/2) \rightarrow 1\)

\[ M(SX_{2n}, Sz_{2n}) \geq \min \{ M(SX_{2n}, SX_{2n}, t/2), M(SX_{2n}, Sz_{2n}, t/2) \} \]

Thus \(\{y_n\}\) is a Cauchy sequence. By completeness of \((X, M, *)\), \(\{y_n\}\) converges (3)

For all \(n \geq n_0\)

Hence \(ASX_{2n} \rightarrow Sz = TSX_{2n}\)

Similarly \(BTX_{2n-1} \rightarrow Tz = ATX_{2n-1}\)

Now put \(x = Sx_{2n}\) and \(y = Tx_{2n-1}\)

\[ M(ASX_{2n}, BTX_{2n-1}, t) \geq \min \{ M(S^2x_{2n}, T^2x_{2n-1}, t), M(ASX_{2n}, S^2x_{2n}, t), M(BTX_{2n-1}, T^2x_{2n-1}, t), M(ASX_{2n}, ASX_{2n}, t), M(ASX_{2n}, ATX_{2n-1}, t) \} \]

Taking limit as \(n \rightarrow \infty\) and using (4) and (5)

We get \(M(Sz, Tz, kt) \geq M(Sz, Tz, t)\), which implies

\[ Sz = Tz \]

(6)

Now put \(x = y\) and \(y = Tx_{2n-1}\)

\[ M(AY, BTTY_{2n-1}, t) \geq \min \{ M(Sy_1, T^2x_{2n-1}, t), M(Ay, Sy_1, t), M(BTTY_{2n-1}, Sy_1, t), M(Ay, T^2x_{2n-1}, t), M(Ty, ATX_{2n-1}, t) \} \]

Taking the limit as \(n \rightarrow \infty\) and using (5) and (6) we get

\[ Az = Tz \]

(7)

Now using (6) and (7)

\[ M(Az, Bz, t) \geq \min \{ M(Sz, Tz, t), M(Az, Sz, t), M(Bz, Tz, t), M(Bz, Sz, t), M(Az, Tz, t), M(Az, Tz, t), M(Az, Tz, t), M(Az, Tz, t) \} \]

\[ \geq M(Az, Bz, t) \]

Which implies \(Az = Bz\)

Using (6), (7) and (8)

We get

\[ Az = Bz = Sz = Tz \]

Now

\[ M(Ax_{2n}, Bz, kt) \geq \min \{ M(Sx_{2n}, Tz, t), M(Ax_{2n}, Sx_{2n}, t), (Bz, Tz, t), M(Bz, Sx_{2n}, t), M(Ax_{2n}, Tz, t), M(Tx_{2n}, Az, t) \} \]

Taking the limit as \(n \rightarrow \infty\) and using (9) we get

\[ Z = Bz \]

Thus \(z\) is common fixed point of \(A, B, S, T\).

For uniqueness let \(w\) be another common fixed point then we have

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\[ M(Az, Bw, kt) \geq \min \{ M(Sz, Tw, t), M(Az, Sz, t), M(Bw, Tw, t), M(Bw, Sw, t), M(Az, Tw, t), M(Tz, Aw, t) \} \]

i.e. \[ M(z, w, kt) \geq M(z, w, t) \]
hence \( z = w \) this completes the proof.

III. Conclusion

Here we proved the theorem using the notion of compatibility without exploiting the condition of \( t \)-norm.

References