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A Numerical Integration Scheme For The Dynamic Motion Of Rigid Bodies Using The Euler Parameters

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Abstract
The dynamics of rigid bodies have been studied extensively. However, a certain class of time-integration schemes were not consistent since they added vectors not belonging to the same tangent space (so3), of the Lie group (SO3) of the Special Orthogonal transformations in E3. The work of Cardona[1,2], and later Makinen[3,4], highlighted this fact using the rotation vector as the main parameter in their derivations.
Some other programs in multibody dynamics, such as the work of Haug[5], rely on the Euler parameters, instead of the rotation vector, as the main variable in their formulations. For this class of programs, different time-integration schemes could be used .This paper discusses one such a scheme. As an example of application, the spinning top was used in this paper. For such a problem, the approximate change of the potential energy was found to be an upper bound to the change in the actual total energy during a time step.
***Keywords**:* Rigid body Dynamics, Euler parameters, Lie group, and Numerical Integration.

Plan of the Paper

I. Introduction

Dynamics is an important branch of mechanics that studies the response of bodies in time due to different excitations. Dynamics of rigid bodies is a particular sub-branch, in which the deformation of the body is neglected.

At the core of the study of rigid body dynamics is the rotation matrix of transformation of coordinates. Some authors (for example Simo[6], Cardona [1,2], Argyris[7,8]) classified the rotation matrix as a member of the Lie group of the Special Orthogonal transformations in E3, called SO3. The skew-symmetric matrix, obtained from the corresponding rotation vector, is a member of the tangent space to SO3, namely, the lie algebra so3.

Different time-integration schemes were constructed to study the dynamics of rigid bodies. One drawback in some of these numerical schemes was the consistency of terms appearing in the temporal integration and whether they belonged to the same tangent space. This issue was raised by (Makinen [3,4], Cardona [1,2], Ghosh [9,10]) among others.

In developing their integration schemes, some authors (Cardona [1,2] and Makinen [3,4]) used the rotation vector as the main parameter. However, some programs used in multibody dynamics, like the work of Haug [5], are based on the Euler parameters.

This paper deals with the time-integration scheme of rigid body dynamics using Euler parameters, within the framework of Cardona and Makinen.

We start with a review of Lie group/Lie algebra and the equations of motion used in rigid body dynamics. Then, we summarize the integration scheme used by Cardona and Makinen. This allows us to derive a parallel integration scheme for the Euler parameters.

As an example of application, the spinning top, tackled by several authors (Simo [6], and Makinen [4]), is solved here again using the Euler parameters to show the applicability of the method.

II. Lie group and Lie algebra

One can express the Lie-group **SO3** of the proper orthogonal linear transformations as

$$SO(3) := \{ \mathbf{A} : E^3 \rightarrow E^3 \mid \mathbf{A}^T\mathbf{A} = I, \det \mathbf{A} = 1 \} \tag{2.1}$$

With E^3 being the three-dimensional Euclidean vector space.

The rotation tensor **A**, in three-dimensional dynamics, belongs to the Lie-group.
This rotation tensor can be expressed in terms of the rotation vector **y** using the exponential map, namely,

$$\mathbf{A} = exp(\check{y}) = \mathbf{I} + \check{y} + \check{y}^2/2! + \ldots \tag{2.2}$$

where **I** is the identity matrix and $\check{y}$ is the skew-symmetric tensor formed from the rotation vector **y .**

The skew-symmetric tensors form the Lie-algebra **so3** defined as

$$so(3) := \{ \check{y} : E^3 \rightarrow E^3 \mid \check{y}^T = - \check{y} \} \tag{2.3}$$

One can show that the derivative of the rotation tensor at the identity is the skew-symmetric tensor $\check{y}$. Thus, $\check{y}$ belongs to the tangent space of the rotation manifold with the identity **I** being the base point.

III. Equations of Motion of a Rigid Body in 3D Space

The equation of motion of an unconstrained rigid body in 3D space can be expressed as:

$$J\dot{\omega} + \tilde{\omega} J\omega = m \tag{3.1}$$

where J=moment of inertia of the rigid body about its body coordinate system.
ω=angular velocity of rigid body about its body coordinate system.
m= resultant moment about the center of gravity of the rigid body, in the body coordinate system.

IV. Consistent time-integration scheme based on the rotation vector – Cardona and Makinen's work.

The main point stressed by Makinen [3,4] was that in any time-integration scheme, all terms in the temporal integration equation should belong to the same tangent space. Thus, the incremental rotation vector, angular velocity vector and acceleration vector, which at time steps t_n and t_{n+1} belong to different tangent spaces, cannot be in the same time-integration equation.

The above requirement was implemented by Cardona et al. [1,2]. Using the principle of virtual work, they derived the equation of motion in terms of the angle of rotation.

Using a similar procedure of picking the beginning of the time step as a reference point, we derive the incremental equations for the Euler parameters and their derivatives.

V. Local Integration Scheme using the Euler parameters

In this section, we propose a temporal integration scheme using the Euler parameters.
The Euler parameters are derived from the rotation vector **Ψ** and θ = ||**Ψ**|| , the norm of **Ψ.**
They can be expressed as :

$$\mathbf{p} = [cos(\theta/2) \quad | \quad \Psi .\, \theta^{-1}.sin(\theta/2)]^T \tag{5.1}$$

As shown by Makinen [3,4] and Cardona [1,2], by choosing the initial and incremental rotation vectors to belong to the same tangent space allows their additions. Calculating the Euler parameters, from the above rotation vectors belonging to the same tangent space, allows the application of the Taylor's series expansion on **p**.

Setting $s = t - t_n$, and using the Newmark scheme to integrate **p** and d**p**/dt through time, we get:

$$p_{n+1}^{(i)}(s) = p(0) + s.\dot{p}(0) + \frac{s^2}{2}[(1-2\beta).\ddot{p}\ (0) + 2\beta.\ddot{p}_{n+1}^{(i)}(s)] \tag{5.2}$$

and

$$\dot{p}_{n+1}^{(i)}(s) = \dot{p}(0) + s[(1-\gamma)\,\ddot{p}(0) + \gamma\,\ddot{p}_{n+1}^{(i)}(s)] \tag{5.3}$$

Subtracting the above two equations at iterations (i+1) and (i), we get their increments, similar to the angular rotation in [1-4]:

$$\Delta\dot{p} = \frac{\gamma}{\beta.\Delta t}\Delta p \tag{5.4a}$$

and

$$\Delta\ddot{p} = \frac{1}{\beta.\Delta t^2}\Delta p \tag{5.4b}$$

At the beginning of each time step, the values of **p**(0), d**p**(0)/dt and d^2**p**(0)/dt^2 , used in equations (5.2-3), cannot be assigned arbitrarily. The requirements are as follows:

a. $\mathbf{p}(0^+)= \mathbf{p}(t_n^+-t_n) =[1\ \ 0\ \ 0\ \ 0]^T$ since the rotation matrix, **A**, is calculated from $\mathbf{A}(t_{n+1})= \mathbf{A}(t_n)*\mathbf{A}(\mathbf{p}_{n+1})$.
b. $d\mathbf{p}(0^+)/dt=d\mathbf{p}/dt(t_n^+-t_n) =[0|\ ½*\mathbf{w}(t_n^+)\ ]^T$ that takes into account the continuity of the angular velocity at t_n , i.e., $\mathbf{w}(t_n^-)=\mathbf{w}(t_n^+)$.
c. $d^2\mathbf{p}(0^+)/dt^2=d^2\mathbf{p}/dt^2(t_n^+-t_n)$ has the following expressions, depending on whether there is/isn't a discontinuity in the applied moment at time t_n :
   - i- No discontinuity in moment:
     $d^2\mathbf{p}(0^+)/dt^2=d^2p/dt^2(t_n^+-t_n) =[\ -¼*\|\ \mathbf{w}(t_n^+)\|^2\ \ |\ ½*d\mathbf{w}(t_n^+)/dt\ ]^T$
     $=[\ -¼*\|\ \mathbf{w}(t_n^-)\|^2\ \ |\ ½*d\mathbf{w}(t_n^-)/dt\ ]^T$
   - ii- There is a discontinuity in moment:
     $d^2\mathbf{p}(0^+)/dt^2 = d^2\mathbf{p}/dt^2(t_n^+-t_n) =[\ -¼*\|\ \mathbf{w}(t_n^+)\|^2\ \ |\ ½*\ d\mathbf{w}(t_n^+)/dt\ ]^T$
     $= [\ -¼*\|\ \mathbf{w}(t_n^-)\|^2\ \ |$ from the equation of motion at $t_n^+\ ]^T$

VI. Example: Spinning Top

6a. Theory

For the sake of simplifying the theoretical formulation of this spinning top problem, one can start with the rate of the angular momentum in the global coordinate system [6].
This can be written as:

$$d/dt\ [\mathbf{A}\ \mathbf{j}\ \mathbf{w}] = \mathbf{M} \tag{6a.1}$$

where **A**=rotation matrix.
j=moment of inertia of the body in the body coordinate system.
w=angular velocity in the body coordinate system.
M=resultant moment acting on the body in the Global coordinate system.

Integrating eq.(6a.1) from time t_n to time t_{n+1} gives

$$\mathbf{A}_{n+1}\ \mathbf{J}_{n+1}\ \mathbf{w}_{n+1} -\ \mathbf{A}_n\ \mathbf{J}_n\ \mathbf{w}_n = \int \mathbf{M}\ dt \tag{6a.2}$$

Using the temporal α-integration scheme on the moment term leads to

$$\int \mathbf{M}\ dt\ = [\ \alpha.\mathbf{M}_{n+1} + (1-\alpha).\mathbf{M}_n\ ]\Delta t = [\ \alpha.\ \mathbf{A}_{n+1}\ .\mathbf{m}_{n+1} + (1-\alpha).\mathbf{A}_n\ \mathbf{m}_{n.}\ ]\Delta t \tag{6a.3}$$

Where m= resultant moment acting on the body in the body coordinate system.

When numerically integrating the equations of motion, and since the process is generally iterative, one has to compute a residual R.

$$\mathbf{R}= -\mathbf{A}_{n+1}\,\mathbf{J}_{n+1}\,\mathbf{w}_{n+1} + \mathbf{A}_n\,\mathbf{J}_n\,\mathbf{w}_n + [\,\alpha.\,\mathbf{A}_{n+1}\,.\mathbf{m}_{n+1} + (1-\alpha).\mathbf{A}_n\,\mathbf{m}_{n.}\,]\,\Delta t = 0 \qquad (6a.4)$$

Eq.(6a.4) can be linearized about $\mathbf{p}_{n+1}^{(i)}$ as,

$$\mathbf{R}(\mathbf{p}_{n+1}^{(i+1)}) = \mathbf{R}(\mathbf{p}_{n+1}^{(i)}) + D\mathbf{R}(\mathbf{p}_{n+1}^{(i)}).\Delta\mathbf{p} \qquad (6a.5)$$

where $\mathbf{p}$ is the vector of Euler parameters.

Note that since the rotation vectors $\boldsymbol{\Psi}_{n+1}^{(i+1)}$ and $\boldsymbol{\Psi}_{n+1}^{(i)}$ belong to the same tangent space, the increments can be added. Consequently, the corresponding Euler parameters $\mathbf{p}_{n+1}^{(i+1)}$ and $\mathbf{p}_{n+1}^{(i)}$ could be expanded in a Taylor series format, such as:

$$\mathbf{p}_{n+1}^{(i+1)} = \mathbf{p}_{n+1}^{(i)} + \Delta\mathbf{p}_{n+1}^{(i)}$$

The linear map $D\mathbf{R}(\mathbf{p}_{n+1}^{(i)}).\,\Delta\mathbf{p}$ could be calculated using the Frechet or Gateaux directional derivative, namely:

$$D\mathbf{R}(\mathbf{p}_{n+1}^{(i)}).\,\Delta\mathbf{p} = d/d\varepsilon[\mathbf{R}(\mathbf{p}+\varepsilon.\Delta\mathbf{p})]_{\varepsilon=0} \qquad (6a.6)$$

Using eq.(6a.4), eq.(6a.6) could be written as

$$D\mathbf{R}(\mathbf{p}_{n+1}^{(i)}).\,\Delta\mathbf{p} = -d/d\varepsilon[\mathbf{A}_{n+1}\,(\mathbf{p}+\varepsilon.\,\Delta\mathbf{p})]_{\varepsilon=0}\,\mathbf{J}_{n+1}\,\mathbf{w}_{n+1} - \mathbf{A}_{n+1}\,\mathbf{J}_{n+1}\,d/d\varepsilon[\mathbf{w}_{n+1}\,(\mathbf{p}+\varepsilon.\,\Delta\mathbf{p})]_{\varepsilon=0}$$

$$+\,\alpha\,d/d\varepsilon[\mathbf{A}_{n+1}\,(\mathbf{p}+\varepsilon.\,\Delta\mathbf{p})]_{\varepsilon=0}\,\mathbf{m}_{n+1}\,\Delta t + \alpha\,\mathbf{A}_{n+1}\,d/d\varepsilon[\mathbf{m}_{n+1}\,(\mathbf{p}+\varepsilon.\,\Delta\mathbf{p})]_{\varepsilon=0}\,\Delta t \qquad (6a.7)$$

where

$$d/d\varepsilon[\mathbf{w}_{n+1}\,(\mathbf{p}+\varepsilon.\,\Delta\mathbf{p})]_{\varepsilon=0} = d/d\varepsilon[2\mathbf{G^*}d\mathbf{p}_{n+1}/dt]\,(\mathbf{p}+\varepsilon.\,\Delta\mathbf{p})_{\varepsilon=0}$$

and

$$d/d\varepsilon[\mathbf{m}_{n+1}\,(\mathbf{p}+\varepsilon.\,\Delta\mathbf{p})]_{\varepsilon=0} = d/d\varepsilon[\mathbf{XL}^{\sim}\,\mathbf{A}_{n+1}^{T}\,[0\ 0\ 1]^{T}(-mg)]_{\varepsilon=0}$$

where $\mathbf{w} = 2\mathbf{G^*}d\mathbf{p}/dt$, m = mass of the spinning top, and

$\mathbf{XL}^{\sim}$ = skew-symmetric matrix formed from the moment-arm vector extending from the origin to the center of gravity of the spinning top.

Then, eq.(6a.7) can be expanded as

$$D\mathbf{R}(\mathbf{p}_{n+1}^{(i)}).\,\Delta\mathbf{p} = -d/d\varepsilon[\mathbf{A}_{n+1}\,(\mathbf{p}+\varepsilon.\,\Delta\mathbf{p})]_{\varepsilon=0}\,\mathbf{J}_{n+1}\,.2\mathbf{G}(d\mathbf{p}_{n+1}/dt) - 2\mathbf{A}_{n+1}\,\mathbf{J}_{n+1}\,d/d\varepsilon[\mathbf{G}_{n+1}\,(\mathbf{p}+\varepsilon.\,\Delta\mathbf{p})]_{\varepsilon=0}\,d\mathbf{p}_{n+1}/dt - 2\mathbf{A}_{n+1}\,\mathbf{J}_{n+1}\,\mathbf{G}_{n+1}\,.\Delta\,(d\mathbf{p}_{n+1}/dt) + \alpha\,d/d\varepsilon[\mathbf{A}_{n+1}\,(\mathbf{p}+\varepsilon.\,\Delta\mathbf{p})]_{\varepsilon=0}\,\mathbf{m}_{n+1}\,\Delta t + \alpha\,\mathbf{A}_{n+1}\,\mathbf{XL}^{\sim}\,d/d\varepsilon[\mathbf{A}_{n+1}^{T}\,(\mathbf{p}+\varepsilon.\,\Delta\mathbf{p})]_{\varepsilon=0}\,[0\ 0\ 1]^{T}(-mg)]\,\Delta t \qquad (6a.8)$$

NOTE:

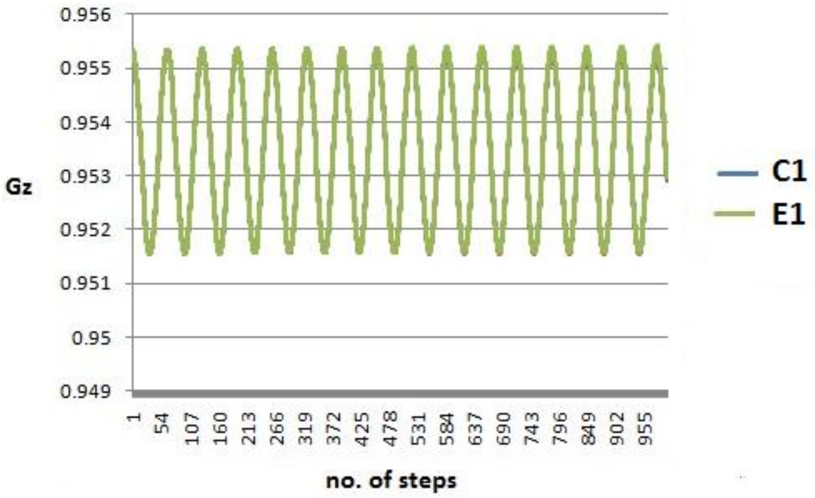
a. In evaluating the product of some of these matrices, and for any matrix $\mathbf{M}$ and vector $\mathbf{v}$, one can transform $d/d\varepsilon[\mathbf{M}\,(\mathbf{p}+\varepsilon.\,\Delta\mathbf{p})]_{\varepsilon=0}.\mathbf{v}$ into $\mathbf{N(v)}.\Delta\mathbf{p}$, where $\mathbf{N(v)}$ is a matrix formed from the vector $\mathbf{v}$.
b. The fourth constraint equation, from the normalization of the Euler parameters, is linearized in a similar manner.
c. The procedure described above is based on starting with a new reference for the value of the Euler parameters at each time step.
   Thus, $A_{(n+1)}^{(i+1)} = A_{(n)} * A(p_{(n+1)}^{(i+1)})$. This is referred to as solution (E1).

6b. Solution

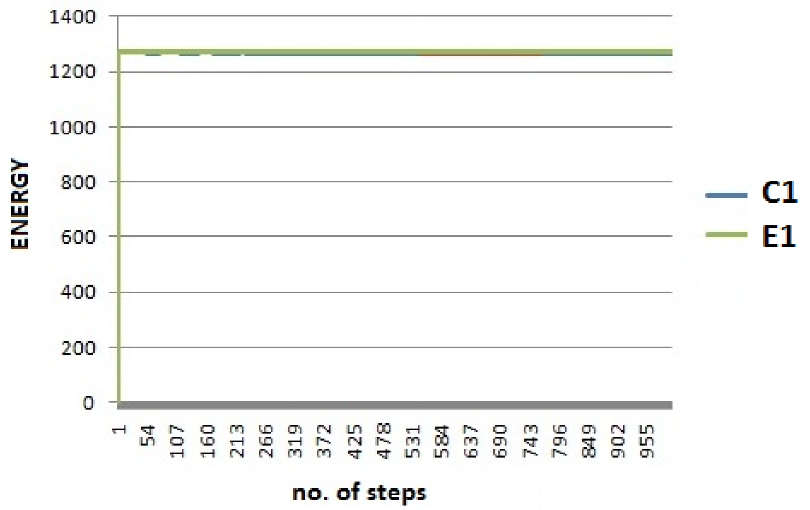
The spinning top, solved in [4,6], is used to validate the above procedure using two different methods, namely:

1. Cardona-Makinen's method of using the rotation vector as the integration parameter. Referred to as method (C1).
2. Using the Euler parameters with one single origin at each time step. Referred to as method (E1).

The results obtained for the above two cases are shown below. As evident, they are close.



(a)



(b)

Fig. 1 (a) Vertical position of the center of gravity of spinning top versus the number of iterations.
(b) The energy of the spinning top versus the number of iterations.

The relative error, using the above integration scheme, for the total energy, angular frequencies of nutation and precession is shown in Figure 2 below for the time steps Δt=0.001 and Δt=0.0001, respectively. Agreement with the closed form solution is good.

Since the conservation of the total energy is not enforced a priori in this algorithm, it was observed that the change in estimated potential energy, using the Euler parameters and their derivatives at the beginning of the time step, provided an upper bound to the change in the actual total energy within a time step.

However, this was not the case for the approximated kinetic energy, which actually did not provide a meaningful upper bound when all the terms were included, and oscillated when the terms included were reduced.

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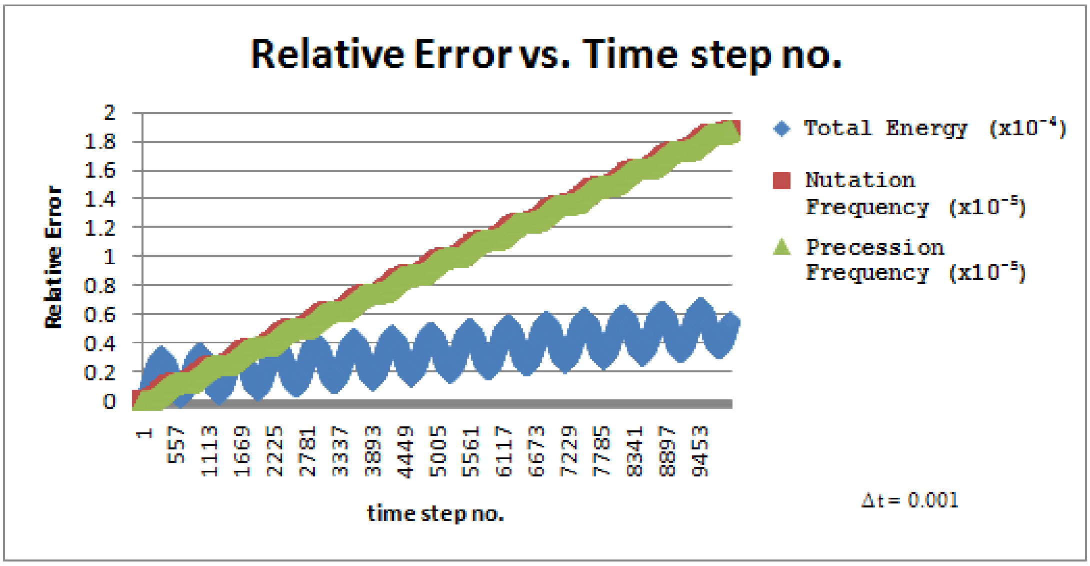
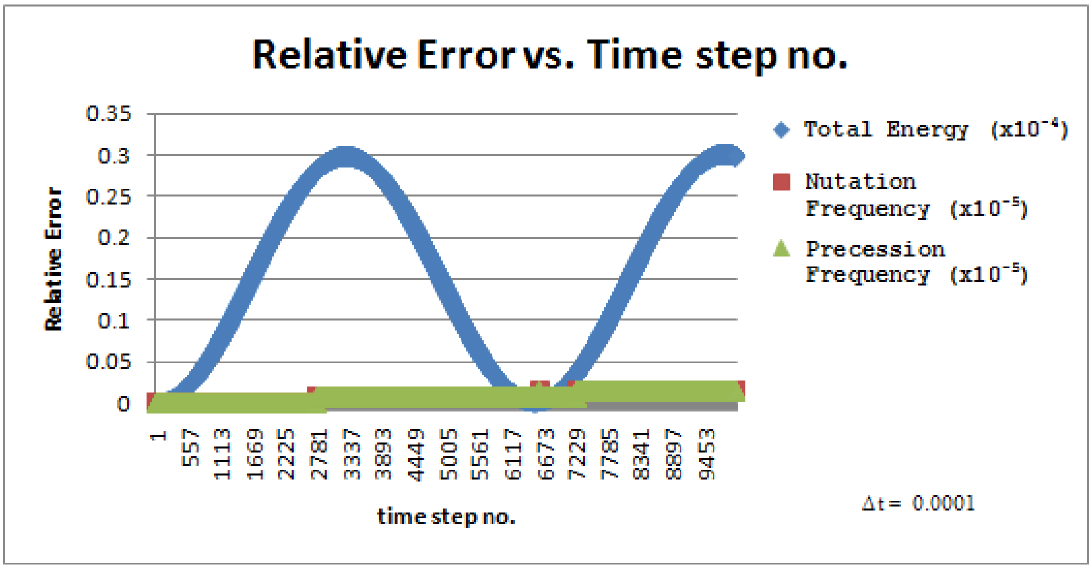


(a)



(b)

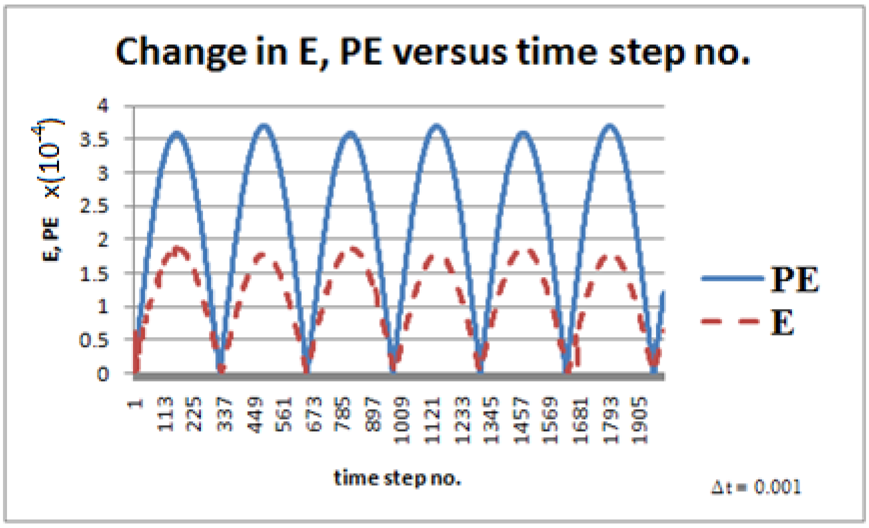
Fig. 2 Relative Error in the total energy, angular frequencies of nutation and precssion versus time step no. for: **(a)** Δt=0.001 , **(b)** Δt=0.0001

This procedure could be explained as follows:
Let pe_0 be the potential energy at the beginning of the time step. The potential energy at the end of the time step n+1 can be expressed as pe($\mathbf{p}_{n+1}$), where $\mathbf{p}_{n+1}$ are the Euler parameters at the end of the time step.
Now, using eq. (5.2) in the definition of potential energy, and replacing d^2p_{n+1}/dp^2 by d^2p_n/dp^2, namely,

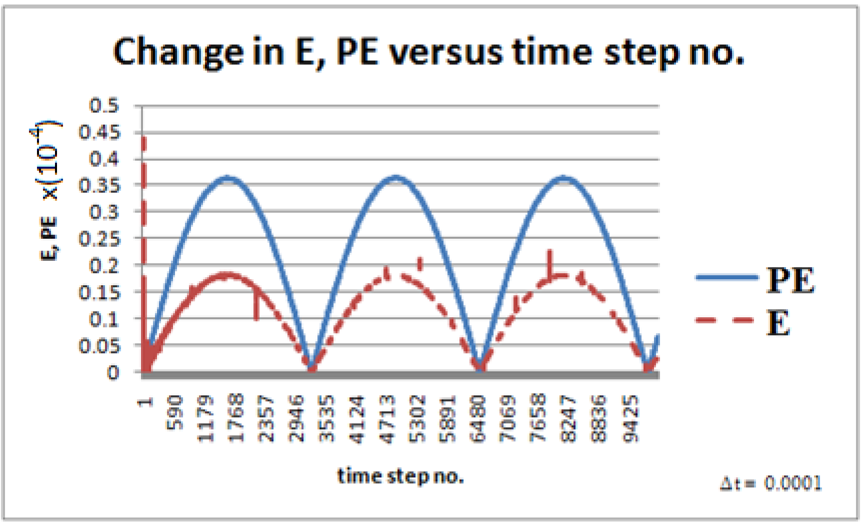
$$p_{n+1}^{(approximate)}(s) = p(0) + s.\dot{p}(0) + \frac{s^2}{2}[(1-2\beta).\ddot{p}\ (0) + 2\beta.\ddot{p}\ (0)] \qquad (6b.1)$$

we get an estimate on the change of potential energy (pe-pe_0).

Plots of the change in the estimated potential energy (using Euler parameters and their derivatives at the beginning of the time step) and the actual total energy change, over a time step, versus the time step number, are shown below in Figure 3 for the time steps Δt=0.001 and Δt=0.0001, respectively.



(a)



(b)

Fig. 3 Change in the estimated potential energy and the actual total energy versus time step no. for: **(a)** Δt=0.001 , **(b)** Δt=0.0001

Indeed, the estimated change in the potential energy provided a meaningful upper bound to the actual change in the total energy for a given time step.

VII. Conclusions

Consistent numerical integration schemes for rigid body dynamics, discussed by Cardona [1,2] and Makinen [3,4] using the rotation vector, were validated based on results from Lie group and Lie Algebra. A numerical integration scheme based on such idea for Euler parameters is discussed in this paper. The spinning top problem, solved in [4,6], was used as an application example. The approximate potential energy was found to be an upper bound to the change in the actual total energy within a time step.

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