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Nonlinear Algebraic Systems with Three Unknown Variables

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ABSTRACT: *Work is devoted to the study of nonlinear algebraic systems with three unknowns and is a continuation of the author's research in this area. Previously, for nonlinear algebraic of system of equations was built analog determinant of Kramer and it is given a necessary and sufficient condition for the existence of solutions of nonlinear algebraic systems with a complex dependence on the variables. For two-parameter systems in the abstract case, it is obtained the results to determine the conditions for finding the number of solutions.*

In this paper describes the method of determining the conditions on the coefficients of algebraic systems to establish the number of solutions of the algebraic system of equations, when an unknowns part in the system as polynomials in any finite degree. This method can be applied to other algebraic systems with more than three unknown variables.

Keywords: *eigen and associated vectors, finite dimensional space, multiparameter system of operators, nonlinear algebraic system of equations, Resultant-operator for two pencils*

I. Introduction

The technique, developed by the author allowed constructing the spectral theory of non-selfadjoint multiparameter system of operators, linear and nonlinear depending on parameter in Hilbert spaces. Considered in this paper the nonlinear algebraic system of equations is a special case multiparameter system when operators are numbers in Hilbert space R. Naturally, that in case of the algebraic systems the got results can be deeper, and methodology less difficult. The proof uses the notion of resultant of two polynomial pencils

Let be the nonlinear algebraic system with three unknown variables.

$$
\begin{aligned}
&a(x,y,z)=a_0+a_1x+\ldots+a_{m_1}x^{m_{1i}}+a_{m_1,}y+\ldots+a_{m_1+n_1}y^{n_1}+a_{m_1+n_1}z=0\\
&b(x,y,z)=b_0+b_1x+\ldots+b_{m_2}x^{m_{2i}}+b_{m_2+1}y+\ldots+b_{m_2+n_2}y^{n_2}+b_{m_2+n_2+1}z+\ldots+b_{m_2+n_2+r_2}z^{r_2}=0 \qquad (1)\\
&c(x,y,z)=c_0+c_1x+\ldots+c_{m_3}x^{m_3}+c_{m_3+1}y+\ldots+c_{m_3+n_3}y^{n_3}+c_{m_3+n_3+1}z+\ldots+c_{m_3+n_3+r_3}z^{r_3}=0
\end{aligned}
$$

In [1,2] for nonlinear algebraic systems with any finite number of unknown variables is given way of constructing an analogue of the determinant of Cramer, in particular, proved: if the continuant of this Cramer ^{i} s determinant is not equal to zero, then the algebraic system has a solution.

In addition [3,4,5], for systems of nonlinear algebraic equations with polynomial dependence on two variables under certain conditions it is defined the number of solutions.

Give some of well-known definitions and concepts necessary for understanding the subsequent presentation.

For two polynomials is introduced the concept of Resultant as follows. Let

$$f(x), g(x);\ \ f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_0,\quad a_n\neq 0;$$

$$g(x)=b_mx^m+b_{m-1}x^{m-1}+\ldots+b_0,\ b_m\neq 0;$$

be two polynomials. Resultant of these polynomials is the operator acting in the space R^{n+m} or C^{n+m} (probably, in some expansion of a field).

$$
\operatorname{Re}s(f,g)=\begin{pmatrix}
a_n & a_{n-1} & \dots & a_1 & a_0 & 0 & \dots & \dots & 0 & 0\\
0 & a_n & \dots & a_2 & a_1 & a_0 & \dots & \dots & 0 & 0\\
. & . & \dots & . & . & . & \dots & \dots & . & .\\
. & . & \dots & . & . & . & \dots & \dots & . & .\\
0 & 0 & \dots & 0 & 0 & a_n & \dots & \dots & a_1 & a_0\\
b_m & b_{m-1} & \dots & b_3 & b_2 & b_1 & b_0 & \dots & 0 & 0\\
0 & b_m & \dots & b_4 & b_3 & b_2 & b_1 & b_0\dots & 0 & 0\\
 & . & \dots & . & . & . & . & \dots & . & .\\
 & . & \dots & . & . & . & . & \dots & . & .\\
 & 0 & 0 & 0 & 0 & b_m & b_{m-1} & \dots & b_1 & b_0
\end{pmatrix}
$$

In a matrix $\operatorname{Re}s(f,g)$ the number of lines with coefficients a_i to equally leading degree of unknown x of a polynomial $g(x)$, that is m, the number of lines of a matrices with numbers b_i coincides with the leading degree of unknown x of a polynomial $f(x)$, that is n. Continuant of this Resultant is a polynomial from coefficients and equal to zero then and only then when polynomials also have a common roots (probably, in some expansion of a field).

The continuant of Resultant can be discovered as a continuant of Sylvester Matrix. Sylvester matrix, allowing calculating continuant of Resultant of two polynomials, is opened by English mathematician James Sylvester.

For a separable polynomial (in particular, for fields of performance zero) this continuant is equal to product of values of one of polynomials on radicals of another (as before, product undertakes in view of a multiplicity of radicals):

The study of such nonlinear algebraic system (1) of equations is spent with help following result from [3]:

let the n bundles depending on the same parameter λ

$$\{B_i(\lambda)=B_{0,i}+\lambda B_{1,i}+\ldots+\lambda^{k_i}B_{k_i,i},\quad i=1,2,\ldots,n$$

$B_i(\lambda)$ - operational bundles acting in a finite dimensional Hilbert space H_i correspondingly. Suppose that $k_1\geq k_2\geq\ldots\geq k_n$. In the space $H^{k_1+k_2}$ (the direct sum of k_1+k_2 tensor product $H=H_1\otimes\ldots\otimes H_n$ of spaces $H_1,H_2,\ldots,H_n$) are introduced the operators $R_i\ \ (i=1,\ldots,n-1)$ with the help of operational matrices (3.12)

Let $B_i(\lambda)$ be the operational bundles acting in a finite dimensional Hilbert space H_i, correspondingly. Without loss ofcopies with

$$
R_{i-1}=\begin{pmatrix}
B^+_{0,1} & B^+_{1,1} & \cdots & B^+_{k_1,1} & \cdots & 0\\
0 & B^+_{0,1} & B^+_{1,1}\cdot\cdot & B^+_{k_1-1.1} & B^+_{k_1,1}\cdot\cdot & 0\\
. & . & \cdot\cdot & . & \cdots & .\\
0 & 0 & \cdot\cdot B^+_{0,1} & B^+_{1,1} & \cdots & B^+_{k_1,1}\\
B^+_{0,i} & B^+_{1,i} & \cdots & B^+_{k_i,i} & 0\cdot\cdot & 0\\
0 & B^+_{0,i} & B^+_{1,i}\cdots & . & B^+_{k_i,i}\cdot\cdot & 0\\
. & . & \cdots & . & \cdots & .\\
0 & 0 & \cdots B^+_{0,i} & B^+_{1,i} & \cdots & B^+_{k_i,i}
\end{pmatrix},\quad i=2,\ldots,n.
$$

The number of lines with operators $B_{s1}, s = 0,1,..., k_1$ in the matrix R_{i-1} is equal to k_2 and the number of lines with operators $B_{si}, s = 0,1,..., k_i$ is equal to k_1. We designate $\sigma_p(B_i(\lambda))$ the set of eigen values of an operator $B_i(\lambda)$.From [3] we have the result:

Theorem 1. $\bigcap_{i=1}^{n}\sigma_p(B_i(\lambda)) \neq \{\theta\}$ then and only then when $\bigcap_{i=1}^{n-1} KerR_i \neq \{\theta\}, (KerB_{k_1} = \{\theta\})$.

In particular, in [5] the two-parameter system operators

$$A(\lambda,\mu)x = (A_0 + \lambda A_1 + \ldots + \lambda^{m_1} A_{m_1} + \mu A_{m_1+1} + \ldots + \mu^{n_1} A_{m_1+n_1})x = 0$$

$$B(\lambda,\mu)y = (B_0 + \lambda B_1 + \ldots + \lambda^{m_2} B_{m_2} + \mu B_{m_2+1} + \ldots + \mu^{n_2} B_{m_2+n_2})y = 0 \quad (2)$$

is studied in tensor product $H_1 \otimes H_2$ of finite-dimensional spaces H_1 and H_2.
Dimension of space H is the product of dimensions of spaces H_1 and H_2. In (2) linear operators $A_i(i = 0,1,\ldots,m_1 + n_1)$ act in finite-dimensional space H_1; and linear operators $B_i(i = 0,1,\ldots,m_2 + n_2)$ act in finite-dimensional space H_2.
If $f_1 \otimes f_2 \in H_1 \otimes H_2$ and $g_1 \otimes g_2 \in H_1 \otimes H_2$ then inner product of these elements in space $H_1 \otimes H_2$ is defined by means of the formulae $[f_1 \otimes f_2, g_1 \otimes g_2]_{H_1\otimes H_2} = (f_1, g_1)_{H_1} \cdot (g_1, g_2)_{H_2}$
This definition is spread to other elements of tensor product spaces on linearity.

Let's reduce a series of known positions concerning the spectral theory of multiparameter system.

Definition1. **[7]** $(\lambda,\mu) \in C^2$ is an eigen value of the system (2) if there are nonzero elements $x \in H_1, y \in H_2$ such that (2) is fulfilled. A tensor $z = x \otimes y$ is named an eigenvector of system (2).

Definition2. [7]. An operator $A_k^+ = A_k \otimes E_2$ (accordingly, $B_k^+ = E_1 \otimes B_k$) where E_1 (accordingly, E_2) is identical operators in H_1(accordingly, H_2), names operator, induced in $H = H_1 \otimes H_2$ by operators A_k (accordingly, B_k).

Definition3.[8] A tensor z_{m_1,m_2} name (m_1, m_2)- the associated vector to an eigenvector $z_{0,0} = x \otimes y$ if following conditions are satisfied

$$\sum_{0\le r_i\le k_i} \frac{1}{r_1!r_2!} \frac{\partial^{r_1+r_2} A^+(\lambda,\mu)}{\partial^{r_1}\lambda\partial^{r_2}\mu} z_{k_1-r_1,k_2-r_2} = 0$$

$$\sum_{0\le r_i\le k_i} \frac{1}{r_1!r_2!} \frac{\partial^{r_1+r_2} A^+(\lambda,\mu)}{\partial^{r_1}\lambda\partial^{r_2}\mu} z_{k_1-r_1,k_2-r_2} = 0$$

$$k_s \le m_s; i = 1,2; s = 1,2.$$

(k_1, k_2) is arrangement from set of the whole nonnegative numbers on 2 with possible recurring and zero.

By means of the approach stated in [5,6], we can establish completeness, multiple completeness of system of eigen and associated vectors, a possibility of multiple expansions on system of eigen and associated vectors of multiparameter system (2).

Theorem2. Let operators $A_i(i = 0,1,\ldots,m_1 + n_1)$ also $B_i(i = 0,1,\ldots,m_2 + n_2)$ act in finite-dimensional spaces H_1 and H_2, accordingly, and one of three following conditions is fulfilled:

a) $\max(m_1n_2, m_2n_1) = m_1n_2$, $KerA_{m_1} = \{\theta\}, KerB_{m_2+n_2} = \{\theta\}$; $A_{m_1}, B_{m_2+n_2}$ are self - conjugate operators everyone in their space

b) $\max(m_1n_2, m_2n_1) = m_2n_1$, $KerB_{m_2} = \{\theta\}, KerA_{m_1+n_1} = \{\theta\}$

$A_{m_1+n_1}$, B_{m_1} - self-conjugate operators, acting in their spaces

c) $m_1n_2 = m_2n_1$, $Ker(A_{m_1}^{n_2} \otimes B_{m_1+n_1}^{n_1} + (-1)^{n_1n_2} A_{m_1+n_1}^{n_2} \otimes B_{m_2}^{n_1})$, $A_{m_1}, B_{m_2}, A_{m_1+n_1}, B_{m_2+n_2}$ are the self-conjugate operators, acting everyone in finite-dimensional space, correspondingly.

Then the $\max(m_1n_2, m_2n_1)$ -fold base on system of eigen and associated vectors of (2) takes place.

At the study of the system (1) we will be used essentially the results from [5,6].:

In (1) we fix variables x, y. Let $x = x_0, y = y_0$ be. Then the system (1) contains three polynomials depending on one variable z. Using the result of the theorem2, we build operators R_1 и R_2.They are the Resultant- operators of polynomials $a(x_0, y_0, z)$ and $b(x_0, y_0, z)$, and also polynomials $a(x_0, y_0, z)$ and $c(x_0, y_0, z)$, correspondingly.

Introduce the notations:

$$\begin{aligned}
\tilde{a}(x,y) &= a_0 + a_1x + \ldots + a_{m_1}x^{m_{1i}} + a_{m_1,}y + \ldots + a_{m_1+n_1}y^{n_1} \\
\tilde{b}(x,y) &= b_0 + b_1x + \ldots + b_{m_2}x^{m_{2i}} + b_{m_2+1}y + \ldots + b_{m_2+n_2}y^{n_2} \\
\tilde{c}(x,y) &= c_0 + c_1x + \ldots + c_{m_3}x^{m_3} + c_{m_3+1}y + \ldots + c_{m_3+n_3}y^{n_3}
\end{aligned} \tag{3}$$

For the polynomials $a(x_0, y_0, z)$ and $b(x_0, y_0, z)$ Resultant R_1 has a form

$$R_1 = R\{a(x_0,y_0,z), b(x_0,y_0,z)\} = \begin{pmatrix} \tilde{a}(x_0,y_0) & a_{m_1+n_1+1} & . & \cdots & . \\ 0 & \tilde{a}(x_0,y_0) & a_{m_1+n_1+1} & \cdots & 0 \\ . & . & . & \cdots & . \\ . & . & . & ..\tilde{a}(x_0,y_0) & a_{m_1+n_1+1} \\ \tilde{b}(x_0,y_0) & b_{m_2+n_2+1} & b_{m_2+n_2+2} & \cdots & b_{m_2+n_2+r_2} \end{pmatrix}$$

(4)

In the matrices of the operator R_1 the number of lines with $\tilde{a}(x_0, y_0), a_{m_1+n_1+1}$ equal to r_2 and the number of lines with $\tilde{b}(x_0, y_0), b_{m_2+n_2+1}, \ldots, b_{m_2+n_2+r_2}$ is equal to1.

By analogy operator R_2 in the space R^{r_3+1} is determined with the help of matrices

$$R_2 = R\{a(x_0,y_0,z), c(x_0,y_0,z)\} = \begin{pmatrix} \tilde{a}_0(x_0,y_0) & a_{m_1+n_1+1} & . & \cdots & . \\ 0 & \tilde{a}_0(x_0,y_0) & a_{m_1+n_1+1} & \cdots & 0 \\ . & . & . & \cdots & . \\ . & . & . & ..\tilde{a}_0(x_0,y_0) & a_{m_1+n_1+1} \\ \tilde{c}(x_0,y_0) & c_{m_3+n_3+1} & c_{m_3+n_3+2} & \cdots & c_{m_3+n_3+r_3} \end{pmatrix} \tag{5}$$

Continuants of Resultants R_1 and R_2 are equal to zero if and only if matrices of operators R_1 and R_2 have nonzero kernels.

So at the expansion of continuants of Resultants R_1 and R_2 we obtain very bulky forms, in obtained equationswe will operate with the leading degrees of variables in R_1 and R_2 . Further we will work only with the members having the leading degrees. We can write the expansions of continuants of Resultants R_1 and R_2 in the forms:

$$\ldots + a_m^{r_2} b_{m_2+n_2+r_2} x_0^{m_1 r_2} + \ldots + a_{m_1+n_1}^{r_2} b_{m_2+n_2+r_2} y_0^{n_1 r_2} - \ldots - a_{m_1+n_1+1}^{r_2} b_{m_2} x_0^{m_2} - \ldots - a_{m_1+n_1+1}^{r_2} b_{m_2+n_2} y_0^{r_2} \quad (6)$$

$$\ldots + a_m^{r_3} c_{m_3+n_3+r_3} x_0^{m_1 r_3} + \ldots + a_{m_1+n_1}^{r_3} c_{m_3+n_3+r_3} y_0^{n_1 r_3} - \ldots - a_{m_1+n_1+1}^{r_3} c_{m_3} x_0^{m_3} - \ldots - a_{m_1+n_1+1}^{r_3} c_{m_3+n_3} y_0^{r_3} \quad (7)$$

(6) is the expansion of the continuant of the Resultant of the polynomials $a(x_0, y_0, z)$ and $b(x_0, y_0, z)$ from (1),when $x = x_0$ and $y = y_0$.

Expression (7) is the expansion of the continuant of the Resultant –operator of polynomials $a(x_0, y_0, z)$ and $c(x_0, y_0, z)$.

If the couple of numbles(x_1, y_1) turns expression (6) to zero, then from the definition of Resultant follows that at these meanings of variables $x = x_1$ and $y = y_1$ $a(x, y, z_1) = 0$ and $b(x, y, z_1) = 0$ .Thus the first and second equations from (1) have the common decision $z_1(x_1, y_1)$.

If the couple (x_2, y_2) turns (7) to zero, then from the definition of Resultant follows that at these meanings of variables $x = x_2$ and $y = y_2$ we have $a(x, y, z_2) = 0$ and $c(x, y, z_2) = 0$ .Thus the first and third equations from (1) have the common decision $z_1(x_2, y_2)$.

Now we consider the system

$$:\ldots + a_m^{r_2} b_{m_2+n_2+r_2} x^{m_1 r_2} + \ldots + a_{m_1+n_1}^{r_2} b_{m_2+n_2+r_2} y^{n_1 r_2} - \ldots - a_{m_1+n_1+1}^{r_2} b_{m_2} x^{m_2} - \ldots - a_{m_1+n_1+1}^{r_2} b_{m_2+n_2} y^{n_2} = 0$$

$$\ldots + a_m^{r_3} c_{m_3+n_3+r_3} x^{m_1 r_3} + \ldots + a_{m_1+n_1}^{r_3} c_{m_3+n_3+r_3} y^{n_1 r_3} - \ldots - a_{m_1+n_1+1}^{r_3} c_{m_3} x^{m_3} - \ldots - a_{m_1+n_1+1}^{r_3} c_{m_3+n_3} y^{n_3} = 0 \quad , \quad (8)$$

For any solution (x, y) of the system (8) the common solution of $a(x, y, z_1) = 0$ и $b(x, y, z_1) = 0$ denote $z_1(x, y)$ and the common solution of $a(x, y, z_2) = 0$ and $c(x, y, z_2) = 0$ denote $z_2(x, y)$. So variable z enter the equation $a(x, y, z) = 0$ linearly we have $z_1(x, y) = z_2(x, y) = z(x, y)$. Consequently, any solution (x, y) of (8) corresponds to only one solution (x, y, z) of the system (1), i.e. the number of solutions of (1) equals the number of solutions of a nonlinear algebraic system (8) with two unknowns.

System (8) is a special case of (2), studied by the author in [5]. In (8) the real numbers a_i, b_j, c_k $(i = 1,2,..., m_1 + n_1 + 1;\ j = 1,2,..., m_2 + n_2 + r_2;\ k = 1,2,..., m_3 + n_3 + r_3)$ are operators in the space R , variables x, y act as parameters, the eigenvectors of the system (8) is constant

Let $m_1 r_2 > m_{2;}, \ m_1 r_3 > m_3, \ n_1 r_2 > n_2, \ n_1 r_3 > n_3$ be then the system (8) is written in the form

$$\ldots + a_{m_1}^{r_2} b_{m_2+n_2+r_2} x^{m_1 r_2} + \ldots + a_{m_1+n_1}^{r_2} b_{m_2+n_2+r_2} y^{n_1 r_2} = 0$$

$$\ldots + a_{m_1}^{r_3} c_{m_3+n_3+r_3} x^{m_1 r_3} + \ldots + a_{m_1+n_1}^{r_3} c_{m_3+n_3+r_3} y^{n_1 r_3} = 0 \quad (9)$$

Theorem3. Suppose the following conditions satisfy:

1) $m_1r_2 > m_{2;}, \quad m_1r_3 > m_3, \quad n_1r_2 > n_2, \quad n_1r_3 > n_3$ (10)

2) $(a_{m_1}^{r_3} b_{m_2+n_2+r_2})^{n_1r_3} (a^{r_3}{}_{m_1+n_1} c_{m_3+n_3+r_3})^{n_1r_2} + (-1)^{n_1^2 r_1 r_2} (a_{m_1+n_1}^{r_2} b_{m_2+n_2+r_2})^{n_1r_3} (a_{m_1}^{r_3} c_{m_3+n_3+r_3})^{n_1r_3} \neq 0$

Then the system (1) has $m_1n_1r_1r_2$ solutions.

Using this approach, we can consider different cases (for example: $m_1r_2 > m_{2;}, \quad m_1r_3 > m_3,$ $n_1r_2 > n_2, \quad n_1r_3 > n_3$ and so on)

II. Conclusion

This paper presents a method of determining the number of solutions of nonlinear algebraic equations. Particularly, it is considered the nonlinear algebraic system of equations with three unknowns.
Since the proof of Theorem 3 we are faced with an unwieldy calculations we restrict ourselves to a single nonlinear algebraic system.
. If necessary, this method will allow to reader to determine the number of solutions of the interested algebraic system.

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