Criterion of functional fullness in many-valued logic

Maydim A. Malkov

Russian Research Center for Artificial Intelligence

Mon Nov 03 23:21:12 2014

Using Slupecki’s criterion of functional fullness we give very simple proof of Jablonskij’s criterion. Then using Jablonskij’s criterion we get criterion of minimal closed sets: a subset of functions of \( P_k \) (k2) is full in \( P_k \), if it contains a non-fictitious two-ary function taking all values and functions of minimal closed sets of some type. The criterion is very useful. In particular, Slupecki’s criterion uses 27 functions, Jablonskij’s criterion uses 21 functions, the criterion of minimal closed sets uses 3 functions in 3-valued logic. The criteria use 255, 231, 6 functions in 4-valued logic respectively and 3125, 3005, 10 functions in 5-valued logic.

I. Introduction

The problem of functional fullness in many-valued logic is very actual and there are several criteria of the fullness (a set \( P_k \) of functions of \( k \)-valued logic is functional fullness if the set generates all functions of the logic by using only composition of functions). Slupecki’s criterion [1] is more popular. By this criterion, a set of functions is full if the set contains any non-fictitious (without fictitious variables) two-ary function taking all values from 0 to \( k - 1 \) in \( k \)-valued logic and if the set contains all one-ary functions.

But Jablonskij’s criterion [2] is more useful. By this criterion, a set of functions is full if the set contains any non-fictitious two-ary function taking all values and all one-ary functions taking not all values. Unfortunately the criterion is not popular. Probably, the criterion is not popular since its proof is very complex. But we have found very simple proof of the criterion. The proof is given in section 2.

Using Jablonski’s criterion we have found the more useful criterion of the functional fullness. By this criterion, a set of function is full if the set contains any non-fictitious two-ary function taking all values and one-ary functions of closed minimal sets of some type (a set is closed if it generates only its functions by compositions, and a set is minimal closed if it does not contain other closed sets). In particular, Slupecki’s criterion uses 27 functions, Jablonskij’s criterion uses 21 functions, the criterion of minimal closed sets uses 3 functions in 3-valued logic. The criteria use 255, 231, 6 functions in 4-valued logic respectively and 3125, 3005, 10 functions in 5-valued logic. The corresponding theorems are main in the article.

Except introduction, the article has 4 sections. Section 2 contains very simple proof of Jablonskij’s criterion using Slupecki’s criterion. Section 3 contains the new proof of number of minimal closed sets. Section 4 gives definitions of families of functions that will be used to prove the main theorems. The definitions are extensions of Post definitions. And section 5 gives the proof of main theorems.

Below \( k \) is only a value of logic, \( P_k \) is a set of all functions of \( k \)-valued logic, and subscript \( i \) is used for any member of a set or for all members of a set.

II. Jablonskij’s criterion

Jablonskij proved his theorem without Slupecki’s criterion. We will prove his theorem by using Slupecki’s criterion.

Theorem A subset of functions of \( P_k \) (k3) is full in \( P_k \), if it contains any non-fictitious two-ary function \( f \) taking all values and all one-ary functions taking not all values.

Proof. Let \( f_0 \) be any one-ary function taking all values. We must prove that there are one-ary functions \( f_1 \) and \( f_2 \) such that they take not all values and \( f(f_1(x), f_2(x)) = f_0(x) \).

Let a sequence \( (f(j_0, j'_0),..., f(j_{k-1}, j'_{k-1})) \) be such that \( f(j_i, j'_i) = f_0(i) \) (there are very much such sequences). Then the set of all \( j_i \) and the set of all \( j'_i \) contain not all values. Indeed, if every set of \( j_i \) contains all values then the first variable of \( f \) is fictitious but \( f \) cannot be fictitious. Then if values of \( f_i \) are
(\(j_0,\ldots, j_{k-1}\)) and values of \(f_2\) are \((j'_0,\ldots, j'_{k-1})\) for values \((0,\ldots, k-1)\) of \(x\) then \(f(f_1(x), f_2(x)) = f_0(x)\).

### III. Minimal closed sets

Every minimal closed set contains only one member, which is a constant or a non-fictitious function. The function is a unit of some symmetric group or some monoid. Fictitious functions are absent in minimal closed sets since any closed set of fictitious function contains a closed set of a constant.

There is a new proof of the theorem of [3].

**Theorem (Lau, 2006)** The number of minimal closed sets equals \(\sum_{m=1}^{k} \binom{k}{m} m^{k-m}\).

**Proof.** One-ary functions will be named just functions.

Let a function take \(m\) values from 0 to \(m-1\). The function will be unit of a monoid, if it is:

\[
\begin{pmatrix}
0,\ldots, m-1, & m, & \ldots, & k-1 \\
0,\ldots, m-1, & j_m, & \ldots, & j_{k-1}
\end{pmatrix}
\]

where \(j \neq i\) and \(j \in \{0,\ldots, m-1\}\). The function is a unit since

\[
\begin{pmatrix}
0,\ldots, m-1, & m, & \ldots, & k-1 \\
0,\ldots, m-1, & j_m, & \ldots, & j_{k-1}
\end{pmatrix} = \begin{pmatrix}
0,\ldots, m-1, & m, & \ldots, & k-1 \\
0,\ldots, m-1, & j_m, & \ldots, & j_{k-1}
\end{pmatrix}
\]

where all \(j_i\) are members of \(\{0,\ldots, m-1\}\).

The number of such functions equals \(m^{k-m}\) since the sequence of \((j_m,\ldots, j_{k-1})\) has values from \(0,\ldots,0\) (\(k-m\) times) to \(m-1,\ldots, m-1\) (\(k-m\) times).

The number of all functions getting \(m\) arbitrary values (not only from 0 to \(m-1\)) equals \(\binom{k}{m}\).

Hence the number of units getting \(m\) values equals \(\binom{k}{m} m^{k-m}\). The number of all units equals

\[
\sum_{m=1}^{k} \binom{k}{m} m^{k-m}.
\]

### IV. Families of functions with the same diagonals

We will use the families to prove the main theorem.

Further, families of functions with the same diagonals are called briefly families.

Families of functions were introduced by Post. There are 4 families in two-valued logic: \(\alpha, \beta, \gamma, \delta\).

Families of functions exist in all \(k\)-valued logics, too. The number of these families is \(2^k\).

The family \(\alpha\) of a function \(f\) is defined standardly: \(f(x,\ldots, x) = x\). The family \(\delta\) is defined standardly, too: \(f(x,\ldots, x) \neq x\). Definitions of the other families are special in the \(k\)-valued logics but they are an extension of Post’s definitions.

We will use new names of families such that the names are subsets of \(\{0,\ldots, k-1\}\).

**Definition** The name of a family of a function \(f\) is the set \(\{j_0,\ldots, j_{m-1}\}\), where \(m \leq k, j_i\) mense \(j\) with subscript \(i\). \(f(x,\ldots, x) = x\) if \(x = j_0 \vee \ldots \vee x = j_{m-1}\) and \(f(x,\ldots, x) \neq x\) if \(x \neq j_0 \land \ldots \land x \neq j_{m-1}\).

In particular, the \(\alpha\) family has name \(\{0,1,\ldots, k-1\}\). the \(\delta\) family has name \(\{\}\). In two-valued
logic, the $\beta$ family has name $\{1\}$, the $\gamma$ family has name $\{0\}$.

Let $i$ be a member of a family name. Then any function $f$ of the family has value $i$, if values of all variables of the function are $i: f(i,\ldots,i) = i$.

A constant $i$ belongs to family $\{i\}$.

**Definition** A family has order $m$ if its name has $m$ members.

V. **Criterion of minimal closed sets**

5.1. Criterion for all minimal closed sets

It is better to prove at first the criterion for all minimal closed sets and then to prove more detail criteria.

**Theorem** A subset of functions of $P_k$ ($k \geq 2$) is full in $P_k$, if it contains a non-fictitious two-ary function taking all values and if it contains all functions of minimal closed sets.

**Proof.** It is enough to prove that functions of minimal closed sets generate all one-ary functions taking not all values since we use Jablonskij’s theorem. As a rule, we will name one-ary functions just functions.

We will use the rule of inverse induction. In the first part of proof we will prove that functions of minimal closed sets generate all functions of families taking not all values and belonging to family of order $k - 1$. We suppose that all functions taking not all values and belonging to families of order more than $m$ are generated by functions of minimal closed sets. In the second part we will prove that some families of order more than $m$ generate all functions taking not all values and belonging to families of order great than or equal to $m$.

1. Any function of family of order $k - 1$ is

$$\begin{align*}
0,\ldots,m' - 1, m', m' + 1,\ldots,k - 1 \\
0,\ldots,m' - 1, i', m' + 1,\ldots,k - 1
\end{align*}$$

for some $m'$ and $i'$. If $i' \neq k - 1$ then the function takes not all values and the function belongs to minimal closed set.

2. Let a function of family $\{0,\ldots,m - 1\}$ of order $m < k - 1$ be

$$\begin{align*}
0,\ldots,m - 1, m,\ldots,k - 1 \\
0,\ldots,m - 1, j_m,\ldots,j_{k - 1}
\end{align*}$$

where all $j_i \neq i$, some $j_i < m$ and (or) some $j_i$ equal. The function takes not all values. We must prove that this function can be generated.

It is enough to prove for one $j_i < m$ and for only two equal $j_i$.

Let some $j_i < m$. Then

$$\begin{align*}
0,\ldots, i_1 - 1, i_1, i_1 + 1,\ldots,k - 1 \\
0,\ldots, i_1 - 1, j_{i_1}, i_1 + 1,\ldots,k - 1
\end{align*}$$

$$\begin{align*}
0,\ldots,m - 1, m,\ldots,i_1 - 1, i_1, i_1 + 1,\ldots,k - 1 \\
0,\ldots,m - 1, j_m,\ldots,j_{i_1} - 1, i_1, j_{i_1 + 1},\ldots,j_{k - 1}
\end{align*} =$$

$$\begin{align*}
0,\ldots,m - 1, m,\ldots,i_1, k - 1 \\
0,\ldots,m - 1, j_m,\ldots,j_{i_1}, j_{k - 1}
\end{align*}$$

The first factor is a function of family $\{0,\ldots,i_1 - 1, i_1 + 1,\ldots,k - 1\}$ of order $k - 1$. The second factor is a function of family $\{0,\ldots,m - 1, i_1\}$ of order $m + 1$. The result is a function of order $m$ and the function takes not all values.

Let some two $j_i$ be equal and these $j_i$ $m$. Then
Criterion of functional fullness in many-valued logic

\[
\begin{pmatrix}
0,\ldots,i_1-1,i_1,i_1+1,\ldots,k-1 \\
0,\ldots,i_1-1,j_1,i_1+1,\ldots,k-1
\end{pmatrix}.
\]

\[
\begin{pmatrix}
0,\ldots,m-1,m,\ldots,j_h-1,j_h,j_h+1,\ldots,k-1 \\
0,\ldots,m-1,j_m,\ldots,j_{h-1},j_h,j_{h+1},\ldots,j_{k-1}
\end{pmatrix}
\]

= \[
\begin{pmatrix}
0,\ldots,m-1,m,\ldots,i_i,\ldots,k-1 \\
0,\ldots,m-1,j_m,\ldots,j_h,\ldots,j_{k-1}
\end{pmatrix}
\]

Again the first factor is a function of family \(\{0,\ldots,i_1-1,i_1+1,\ldots,k-1\}\) of order \(k-1\), the second factor is a function of family \(\{0,\ldots,m-1,i_i\}\) of order \(m+1\), and the result takes not all values.

By analogy we can prove the theorem for the other families of order \(m\).

5.2. Criterion for minimal closed sets of families of order \(k-1\)

The number of the sets equals \(k(k-1)\). This number is very small.

**Theorem** Let functions belong to minimal closed sets. Then functions of families of order \(k-1\) generate the other functions.

**Proof.** We use the inverse induction. Functions of families of order \(k-1\) generate themselves. We must prove that functions of families of order \(n+1\) and more can generate any function of family of order \(n\).

Let
\[
\begin{pmatrix}
0,\ldots,n-1,n,\ldots,k-1 \\
0,\ldots,n-1,j_n,\ldots,j_{k-1}
\end{pmatrix}
\]

be a function of family of order \(n\), \(j_i < n\), all \(j_i\) be different, and some \(j_m = n-1\). The function is generated by two functions, one is of family of order \(k-1\) and one is of family of order \(n+1\):

\[
\begin{pmatrix}
0,\ldots,n-1,n,\ldots,m-1,m,m+1,\ldots,k-1 \\
0,\ldots,n-1,n,\ldots,m-1,n-1,m+1,\ldots,k-1
\end{pmatrix}
\]

\[
\begin{pmatrix}
0,\ldots,n-1,n,\ldots,m-1,m,m+1,\ldots,k-1 \\
0,\ldots,n-1,j_n,\ldots,j_{m-1},m,j_{m+1},\ldots,j_{k-1}
\end{pmatrix}
\]

= \[
\begin{pmatrix}
0,\ldots,n-1,n,\ldots,m-1,m,m+1,\ldots,k-1 \\
0,\ldots,n-1,j_n,\ldots,j_{m-1},n-1,j_{m+1},\ldots,j_{k-1}
\end{pmatrix}
\]

If \(j_i\) are not different then the second row of the factor has more than two equal components and we have the same result.

We have the same result too, if a function belongs to a family of order \(n\) but its binomial is other.

5.3. Other criteria

There are many other criteria of functional fullness but all of them use \(k(k-1)/2\) minimal closed sets. But criteria using less number of minimal closed sets do not exist.

The criteria use types of functions. The types were introduced by Post [4] and generalized by us [5,6] for 2-valued logic but they are extended to more valued logic without changes. A function has a type \((F,a^\mu)\), where \(F\) is the family of the function and \(a^\mu\) is Post’s condition:
Criterion of functional fullness in many-valued logic

- if the function has \( m \) lines, which have a column of variables with values only \( a \), and has no \( m+1 \) lines, which have a column of variables with values only \( a \), then \( \mu = m \).
- if the function has \( m \) lines, which have a column of variables with values only \( a \), and has no \( m+1 \) lines, which have value \( a \), then \( \mu = \omega + m \) (including \( m = 0 \)).

Functions of minimal closed sets have \( \mu \in \{0, \omega, \omega + 1\} \), the other \( \mu \) are absent.

We will give a criterion only for minimal functions (a function of a set is minimal if it has minimal arity and the ordinal number, which is contents of column with function values by reading from the first line to the last line, for example” the one-ary function of 3-valued logic with values (0,1,2) has the number \( 5 = 0 \cdot 3^2 + 1 \cdot 3^1 + 2 \cdot 3^0 \)). The criterion for a set of functions is next:
- any function has only one \( a \) with \( a^0 \) and only one \( a \) with \( a^\omega \).
- if any function has \( a^0 \) at \( a = 0 \) then \( a^\omega_0 \) is only for \( a_0 \in \{1,...,k-2\} \).
- if any function has \( a^0 \) at \( 1 \), \( a_1 \), \( k-2 \) then \( a^\omega_0 \) is only for \( a_0 \in \{a+1,...,k-1\} \).
- if any function has \( a^0 \) at \( a = k-1 \) then only \( 0^\omega \).

The criterion includes three functions with values (0,0,2) , (0,1,1) , and (2,1,2) in 3-valued logic.

One more criterion includes three (non-minimal) functions with values (0,1,0) , (0,2,2) , and (1,1,2) . The other criteria do not exist. There are 24 criteria in 4-valued logic, every criterion includes 6 functions.

References